Bandwidth Sharing: Objectives and Algorithms (sec. I,II)

KTH –EP2210
Performance Analysis of Communication Networks
Motivation

• Bandwidth sharing is critical to determine user-perceived performance.
  – Fully utilizing the available bandwidth
  – Maintaining ‘fairness’ among concurrent flows

• Optimize TCP for elastic traffic
  – Possible bandwidth-sharing objectives: fairness
  – Distributed flow-control to realize fairness
Outline

• Bandwidth sharing objectives (section I,II)
  – Max-min fairness [ Bertsekas & Gallager ]
  – Proportional fairness [ Kelly ]
  – Potential delay minimization

• Distributed flow control
  – Fixed window control
  – Rate adjustment
Network model

- A set of links \( L = \{l\} \), each has a capacity \( C_l > 0 \)
- A set of routes \( R = \{r\} \),
  - \( l \in r \): route \( r \) goes through link \( l \)
- \( \lambda_r \): bandwidth (or rate) allocation for flow \( r \)
  - \( \{\lambda_r\} \): a feasible bandwidth sharing iff \( \lambda_r \geq 0, \sum_{r \in r} \lambda_r \leq C_l \)

- An example: a linear network (parking lot scenario)

\[
\text{maximize } \sum \lambda_r \Rightarrow \lambda_0 = 0 \text{ and } \lambda_r = 1, r = 1, \ldots, R \quad (C_l = 1) \Rightarrow \sum \lambda_r = L
\]
Max-min fairness (1)

- Maximize the allocation for the most poorly treated flows

\[
\text{maximize } \min \{ \lambda_r \} \\
\text{subject to } \lambda_r \geq 0, \sum_{r \in l} \lambda_r \leq C_l, r \in R
\]

- Propositions:
  - No rates can be increased without decreasing any already smaller rate
  - Every flow has at least one bottleneck link
    - \( l \) is a bottleneck of \( r \) iff

\[
\sum_{l \in r'} \lambda_{l'} = C_l \quad \text{and} \quad \lambda_r = \max \{ \lambda_{l'}, l \in r' \}
\]
Max-min fairness (2)

• Filling procedure
  1. Increase rate for all flows until one link get saturated
  2. Consider only flows not crossing saturated links, go back to step 1

\[
\text{maximize } \min \{\lambda_r\} \Rightarrow \lambda_r = \frac{1}{2}, r = 0,1,..R \Rightarrow \sum \lambda_r = \frac{L+1}{2}
\]
Proportional fairness (1)

• A feasible allocation \( \{\lambda_r\} \) is Proportional fair if it satisfies

\[
\sum_{R} \frac{\lambda'_r - \lambda_r}{\lambda_r} \leq 0
\]

– For all other feasible allocation \( \{\lambda'_r\} \), the sum of proportional rate changes with respect to the optimum is not positive.

• PF maximizes the overall utility of rate allocations assuming each flow has a logarithmic utility function:

\[
\text{maximize} \quad \sum \log \lambda_r \\
\text{subject to} \quad \lambda_r \geq 0, \sum_{r \in l} \lambda_r \leq C_l, r \in R
\]
Proportional fairness (2)

\[ \sum \frac{\lambda'_r - \lambda_r}{\lambda_r} \leq 0 \Rightarrow \text{maximize } \sum \log \lambda_r \]

• **Proof:**

A differentiable function \( f \) is concave if \( (f(y) - f(x)) \leq \nabla f(x)^T (y - x) \)

\( U(r) = \log \lambda_r \) is a concave function

\[ \Rightarrow \log \lambda'_r - \log \lambda_r \leq (\log \lambda_r)' (\lambda'_r - \lambda_r) \Rightarrow \log \lambda'_r - \log \lambda_r \leq (\lambda'_r - \lambda_r) / \lambda_r \]

\[ \Rightarrow \sum (\log \lambda'_r - \log \lambda_r) \leq \sum \frac{\lambda'_r - \lambda_r}{\lambda_r} \]

if \( \sum \frac{\lambda'_r - \lambda_r}{\lambda_r} \leq 0 \)

\[ \Rightarrow \lambda_r = \arg \max \sum \log \lambda_r \]
Proportional fairness (3)

\[
\begin{align*}
\text{maximize} \quad & \sum_{r=0}^{L} \log \lambda_r \\
\text{s.t.} \quad & \lambda_0 + \lambda_r = 1, \quad r = 1, \ldots, L
\end{align*}
\]

\[
\begin{align*}
\left\{ \log \lambda_0 + \sum_{r=1}^{L} \log(1 - \lambda_0) \right\}' = 0 \Rightarrow \text{The optimum}
\end{align*}
\]

\[
\begin{align*}
\lambda_0 &= \frac{1}{1+L} \\
\lambda_r &= \frac{L}{1+L}, \quad r = 1, \ldots, R
\end{align*}
\]

- The overall throughput \( \sum \lambda_r = L - \frac{L-1}{L+1} \)
- Equally sharing network resource \( L\lambda_0 = \lambda_r, \quad r = 1, \ldots, R \)
Potential delay minimization (1)

• Minimize the time delay needed to complete document transfers
  – Potential delay: the reciprocal of the allocated rate $1/\lambda_r$

$$\begin{align*}
\text{minimize} & \quad \sum 1/\lambda_r \\
\text{subject to} & \quad \lambda_r \geq 0, \sum_{r \in l} \lambda_r \leq C_l, r \in R
\end{align*}$$
Potential delay minimization (2)

minimize \( \sum_{r=0}^{L} 1/\lambda_r \)  
s.t. \( \lambda_0 + \lambda_r = 1, r = 1, \ldots, L \)  

\[ \left\{ 1/\lambda_0 + \sum_{r=1}^{L} 1/(1 - \lambda_0) \right\}' = 0 \Rightarrow \text{The optimum} \]

\[ \lambda_0 = \frac{1}{1 + \sqrt{L}} \]
\[ \lambda_r = \frac{\sqrt{L}}{1 + \sqrt{L}}, r = 1, \ldots, R \]

The overall throughput \( \sum \lambda_r = L + (1 - \sqrt{L}) \)
Comparison - parking lot scenario

<table>
<thead>
<tr>
<th>Objective</th>
<th>Allocation for long flow</th>
<th>Allocation for short flows</th>
<th>Overall throughput</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize total throughput</td>
<td>$\max \sum \lambda_r$</td>
<td>$\lambda_0 = 0$</td>
<td>$\sum \lambda_r = L$</td>
</tr>
<tr>
<td>Max-min fairness</td>
<td>$\max \min { \lambda_r }$</td>
<td>$\lambda_0 = 1/2$</td>
<td>$\sum \lambda_r = \frac{L+1}{2}$</td>
</tr>
<tr>
<td>Proportional fairness</td>
<td>$\max \sum \log \lambda_r$</td>
<td>$\lambda_0 = \frac{1}{1+L}$</td>
<td>$\sum \lambda_r = L - \frac{L-1}{L+1}$</td>
</tr>
<tr>
<td>Potential delay minimization</td>
<td>$\min \sum \frac{1}{\lambda_r}$</td>
<td>$\lambda_0 = \frac{1}{1+\sqrt{L}}$</td>
<td>$\sum \lambda_r = L + (1-\sqrt{L})$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
Th_{\text{max}} & \geq Th_{PF} \geq Th_{\text{Min-Delay}} \geq Th_{\text{Max-min}} \\
(\lambda_0 = 0, \lambda_r = 1) & (\lambda_0 = \lambda_r / L) & (\lambda_0 = \lambda_r / \sqrt{L}) & (\lambda_0 = \lambda_r)
\end{align*}
\]

- Penalizing long flows more severely achieves higher overall throughput

2010-10-05
Liping Wang
Weighted shares

- $\phi_r$: weighting factor associated with each route $r$
- **Max-min fairness**: maximize $\min\{\lambda_r / \phi_r\}$
  
  subject to $\lambda_r \geq 0, \sum_{r\in R} \lambda_r \leq C_l, r \in R$
  
  - Filling procedure: the speed of increase the rate along $r$ should equal $\phi_r$

- **Proportional fairness**: maximize $\sum \phi_r \log \lambda_r$
  
  - The aggregate of weighted proportional rate changes with respect to the optimum allocation satisfies $\sum \phi_r \frac{\lambda_{r}^{'} - \lambda_r}{\lambda_r} \leq 0$

- **Potential delay minimization**: minimize $\sum (\phi_r / \lambda_r)$
Comparison - parking lot scenario (weighted)

<table>
<thead>
<tr>
<th>Objective</th>
<th>Allocation for long flow</th>
<th>Allocation for short flows</th>
<th>Overall throughput</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize total throughput</td>
<td>$\max \sum {\lambda_r / \phi_r}$</td>
<td>$\lambda_0 = 0$</td>
<td>$\lambda_r = 1$</td>
</tr>
<tr>
<td>Max-min fairness</td>
<td>$\max \min {\lambda_r / \phi_r}$</td>
<td>$\lambda_0 = 2 / 3$</td>
<td>$\lambda_r = 1/3$</td>
</tr>
<tr>
<td>Proportional fairness</td>
<td>$\max \sum \phi_r \log \lambda_r$</td>
<td>$\lambda_0 = \frac{2}{2 + L}$</td>
<td>$\lambda_r = \frac{L}{2 + L}$</td>
</tr>
<tr>
<td>Potential delay minimization</td>
<td>$\min \sum (\phi_r / \lambda_r)$</td>
<td>$\lambda_0 = \frac{\sqrt{2}}{\sqrt{2} + \sqrt{L}}$</td>
<td>$\lambda_r = \frac{\sqrt{L}}{\sqrt{2} + \sqrt{L}}$</td>
</tr>
</tbody>
</table>

$\lambda_0 = 0, \phi_0 = 2, \{\phi_r = 1, r = 1, \ldots, R\}$

The increase in $\lambda_r$ is approximately proportional to $\phi_r$ only when the number of routes sharing a link is large.
Outline

• Bandwidth sharing objectives (section I,II)
  – Max-min fairness [ Bertsekas & Gallager ]
  – Proportional fairness [ Kelly ]
  – Potential delay minimization

• Distributed flow control
  – Fixed window control
  – Rate adjustment