SF3674 DIFFERENTIAL GEOMETRY, GRADUATE COURSE, FALL 2016, FIRST SET OF HAND IN EXERCISES

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Exercises

The following is the first set of hand in exercises. They should be handed in by October 7 2016.

(1) Geodesics in the hyperbolic plane.
  a) Calculate the Christoffel symbols of $g_{\mathbb{H}} = y^{-2}(dx^2+dy^2)$ defined on $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and find at least one geodesic.
  b) Prove that translations in x are isometries; that dilations $(x, y) \mapsto c(x, y)$, $0 < c \in \mathbb{R}$, are isometries; and that the inversion $(x, y) \mapsto \frac{(x,y)}{x^2+y^2}$ is an isometry.
  c) Prove that isometries map geodesics to geodesics, and use a) and b) to construct all geodesics on $(\mathbb{H}, g_{\mathbb{H}})$.

(2) Exercise 3.21, page 96 of [1].

(3) Let $H^n_\nu(r)$ and $S^n_\nu(r)$ be the pseudohyperbolic space and pseudosphere respectively, cf. Definition 23, p. 110 of O'Neill's book. For a general semi-Riemannian manifold (M, g), let

$$G = \mathrm{Ric} - \frac{1}{2}Sg,$$

where Ric is the Ricci tensor and S is the scalar curvature of (M, g). G is called the *Einstein tensor*. Compute the Einstein tensor of $H^4_1(r)$ and of $S^4_1(r)$.

Note that $S^4_1(r)$ is *de Sitter space*, a Lorentz manifold of interest in general relativity. The universal covering space of $H^4_1(r)$ is called *anti de Sitter* space.

References

[1] Barrett O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.