KTH ROYAL INSTITUTE
OF TECHNOLOGY

Algebraic Geometry with a view towards applications

Sandra Di Rocco, ICTP Trieste

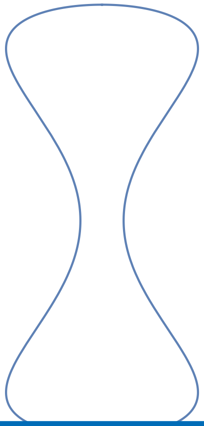
Plan for this course

- Lecture I: Algebraic modelling (Kinematics)
- Lecture II: Sampling algebraic varieties: the reach.
- Lecture III: Projective embeddings and Polar classes (classical theory)
- Lecture IV: The Euclidian Distance Degree
- **Lecture V: Bottleneck degree**

References:

- D. Eklund *the numerical algebraic geometry of bottlenecks*. ArXiv
- DR-Eklund-Weinstein *The bottleneck degree of a variety*. ArXiv

Motivation



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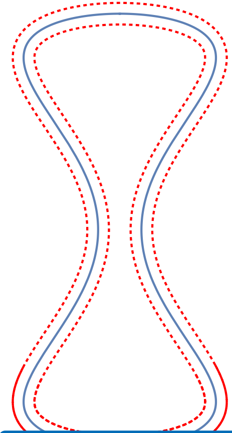

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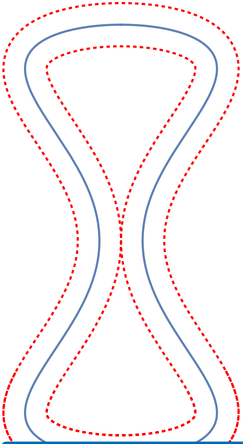

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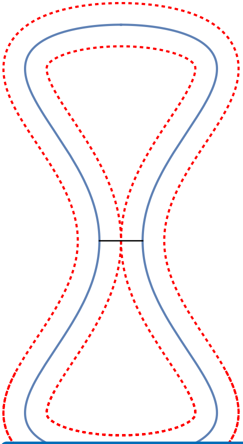

- Consider a growing tubular neighborhood of M.
- At some point it becomes singular ($M \neq \{p\}$).
- Half the black distance is called the *reach* of M.
- And the line is called a *bottleneck*.

Bottlenecks of algebraic varieties

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- For $(x, y) \in X \times Y$ with $x \neq y$, the line joining x and y is a bottleneck if $(x - y) \perp T_x X$ and $(y - x) \perp T_y Y$.

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- Goal: efficient method to compute bottlenecks given defining equations for X and Y.
- Important subgoal: count bottlenecks. Let $\beta(X, Y) :=$ #isolated bottlenecks, $\beta(X) := \beta(X, X)$.

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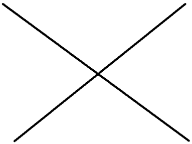
Example

- A pair of generic lines $X, Y \subset \mathbb{C}^2$ has no bottlenecks.



Bottlenecks of algebraic varieties

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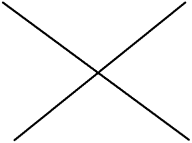
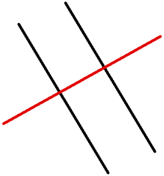
- A pair of generic lines $X, Y \subset \mathbb{C}^2$ has no bottlenecks.
- A pair of (generic) parallel lines in $\mathbb{C}^2$ have a 1-dimensional family of bottlenecks.





Bottlenecks of algebraic varieties

Example

let $X \subseteq \mathbb{C}^2$ be one of the two lines of the isotropic quadric

$$x^2 + y^2 = (x + iy)(x - iy) = 0,$$

say $X = \langle (i, 1) \rangle$.

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- Since $(i, 1) \perp (i, 1)$, X is orthogonal to itself,
- and X has one bottleneck, namely X.

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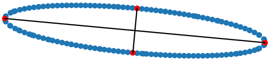
- In the case $X = Y$, there are 2 bottlenecks and there exists a real X with both bottlenecks real.



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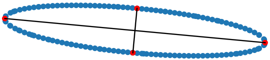

Remark

For generic curves $X, Y \subseteq \mathbb{C}^2$ of degree d_X and d_Y:

$$\beta(X, Y) < d_X^2 d_Y^2 = EDD(X) EDD(Y)$$

Bottlenecks of algebraic varieties

Proposition (D. Eklund 2018)

For smooth curves $X, Y \subseteq \mathbb{C}^n$ in general position of degree d_X and d_Y and genus g_X and g_Y:

$$\beta(X, Y) = (3d_X + 2g_X - 2)(3d_Y + 2g_Y - 2) - |X \cap Y| =$$

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Corollary

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General position: transversal intersection with Q

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- Considering $X = Y$

$$\beta(X, Y) = \text{ isolated bottlenecks of } X$$

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- X and Y intersect Q transversely.

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- As $t : 1 \to 0$ the *start points* $(x, y) \in NL_X(p) \times NL_Y(q)$ follow paths in $\mathbb{C}^n \times \mathbb{C}^n$ to *endpoints* (x, y) such that $\overline{xy}$ is a bottleneck or $x = y$.

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$$\beta(X) = EDD(X)^2 - R = (p_0 + \ldots + p_n)^2 - R$$

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- $X \subset \mathbb{P}^N$ in general position.

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where Δ is the diagonal scheme in $X \times X$.
Consider $g : C \to (\mathbb{P}^2)^\vee = \{ \text{ lines in } \mathbb{P}^2\}$ assigning the line N_pC to p

The bottlenecks can be computed via the associated *Double Point Scheme*

Concrete algebraic formulation of the ideal of bottlenecks and examples worked out in M2
MADDIE later today

Assumption

Q isotropic quadric in $\mathbb{P}^N$.
$X \subset \mathbb{P}^N$ is in general position (BN regular):

- X intersects Q transversally.
- X has only finitely many bottlenecks
- one additional (technical) assumption ...

Theorem (DR-Eklund- Weinstein)

Assume X is in general position, then the number of bottlenecks (counted with multiplicity) is given by explicit polynomials in $P_0, \ldots, P_n, h$ where h id the hyperplane class in $\mathbb{P}^N$

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in particular:

- for a curve C in $\mathbb{P}^2$:
$$\beta(C) = d^4 - 4d^2 + 3d$$
- for a curve C in $\mathbb{P}^3$:
$$\beta(C) = p_1^2 + 2d^2 - 3p_1 - 2d$$
- for a curve S in $\mathbb{P}^5$:
$$\beta(S) = (p_0 + p_1 + p_2)^2 + (p_0 + p_1)^2 + d^2 - \deg(3h^2 + 6hp_1 + 12p_1^2 + p_2)$$

$$p_i = \deg(P_i) = P_i h^{N-i}$$

the affine case

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Theorem (DR-Eklund-Weinstein)

Under the previous assumptions (for $\overline{X}$ and X_∞):

$$\beta(X) = \beta(\overline{X}) - \beta(X_\infty)$$

Polar calculus: example

DR-D. Eklund, C. Peterson. *Adv. App. Math.* 2018

code for p_i, deg $P_i P_j$ and deg$(P_i^k D^m)$ for any divisor D.

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degree	8	12	16	24	32	24	32	32

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$EDD(X) = 40$ and $\beta(X) = ?$

SUMMARY

- **Sampling** can be an efficient method of visualising the variety of solutions of polynomial equations, if we can recover the homology.
- Any sample of size $\varepsilon < \frac{\textbf{reach}}{2}$ recovers the homology of the manifold
- An estimate of the reach requires an estimate of the **bottleneck degree** of the variety.
- The bottleneck can be computed via classical algebraic varieties: **The polar classes.** Generalizing the EDD.

Thanks to Maddie and Sascha

It's been fun!