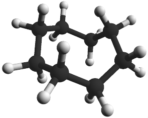


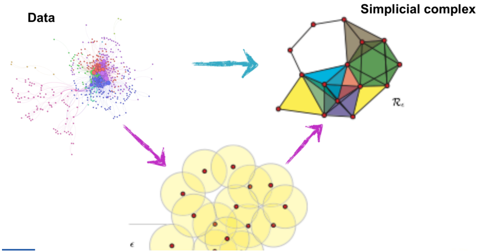
Algebraic Numerics and Sampling:

Algebraic Geometry with a view towards applications

Sandra Di Rocco, ICTP Trieste

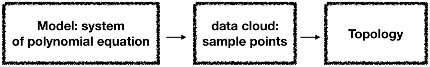
Data analysis

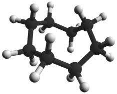
- Given a data cloud, discover the geometry (shape) of the data. (DATA INTERPOLATION)
- Given a data cloud, give it a structure and deduce useful information. (TOPOLOGICAL DATA ANALYSIS)

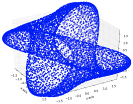


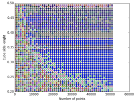
Sampling

We will focus of certain algebraic aspects of data collection coming from algebraic modelling:









Plan for this course

- Lecture I: Algebraic modelling (Kinematics)
- Lecture II: Sampling algebraic varieties: the reach.
- Lecture III: Projective embeddings and Polar classes (classical theory)
- Lecture IV: The Euclidian Distance Degree (closest point)
- Lecture V: Bottleneck degree from classical geometry (back to sampling)

k-revolute serial chain linkage

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a k**R chain** consists ok $(k+1)$ rigid links connected with k revolute joints

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Forward Kinematics Problem (FKP): Compute the position of the end-effector, given the joint-angles

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Inverse Kinematics Problem (IKP): Compute the joint-angles taking the mechanism to a specified position of the end-effector

Rigid Body Motion

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6**R-chain**

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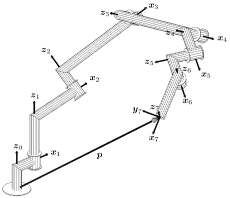

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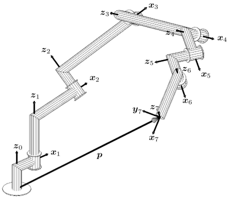

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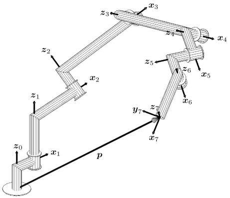

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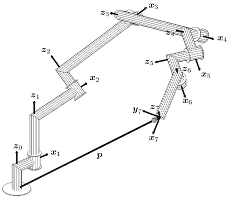

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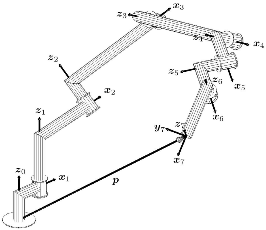

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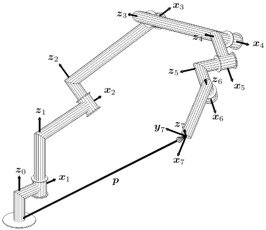

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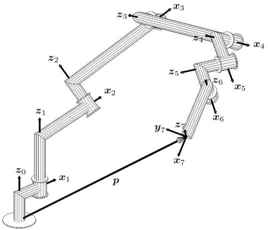

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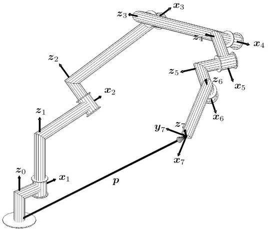

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1988 Li & Liang Efficient algorithm (16 paths)

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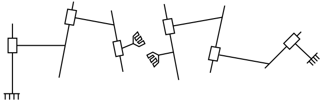

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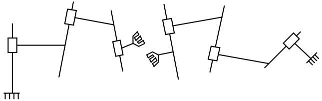

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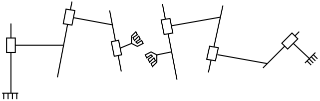

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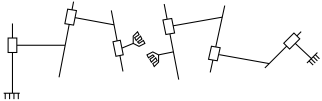

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So the problem is reduced to intersection theory on $\mathcal{Q}$.

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- Can be done f.ex via the BB "plus" and "minus" decompositioncive by a $\mathbb{C}^*$ action.

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- We need to compute $[ImF] \cdot [ImG] = 2ab$.
- Consider the Segre model $F : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{Q}$.

$$F(a_0, a_1, b_0, b_1, c_0, c_1) = \begin{pmatrix} a_0 b_0 c_0 \\ a_1 b_1 c_0 \\ a_1 b_0 c_1 \\ a_0 b_1 c_1 \\ a_1 b_0 c_0 \\ a_0 b_1 c_0 \\ -a_0 b_0 c_1 \\ -a_1 b_1 c_1 \end{pmatrix}.$$

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- Moreover, $ImF \subset \mathbb{P}^7$ has degree $3! = 6$.
- Let $h \in A_5(\mathcal{Q})$ be the hyperplane class. Then
$$6 = \deg(h^3[ImF]) = a\deg(h^3g_1) + b\deg(h^3g_2) = a + b,$$
and hence $b = 2$ and $2ab = 16$.

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How can we compute and present its solution?

Visualization

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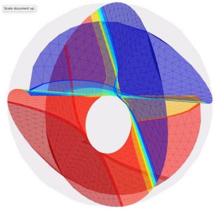
- Currently: Bertini real: D. Brake, Bates-Hauenstein-Sommese-Wampler
- Cell decomposition of almost smooth real algebraic surfaces
  Besana, Di Rocco, Hauenstein, Sommese, Wampler
  Numerical Algorithms (2013)



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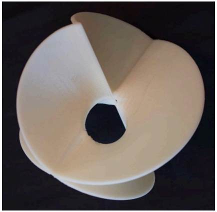

3D printed

Cycloalkane

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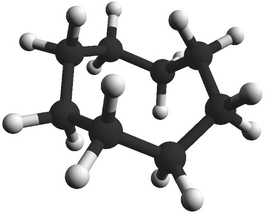

https://commons.wikimedia.org/wiki/File:Cyclooctane_ballandstick.png.

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  - for $i \in \mathbb{Z}_8$, δ_i is the angle between n_i and n_{i+1}.

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- The cyclo-octane surface permits a whole range of symmetries. Consider the transformation

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- there is a component, *the spherical component*, pointwise fixed by this transformation, actually contained in a linear subspace $\mathbb{R}^4 \subset \mathbb{R}^8$.
- *Topology of cyclo-octane energy landscape*, Martin S, Thompson A, Coutsias EA, Watson JP. J Chem Phys. 2010. The surface has two components: one with the homology of a sphere and one with the homology of a Klein bottle.

Sampling a cyclo-octane

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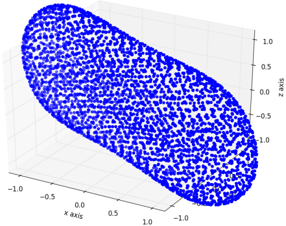

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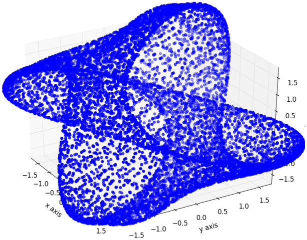

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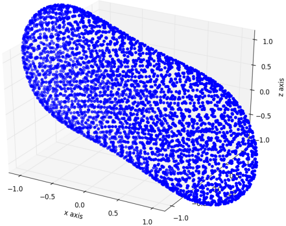

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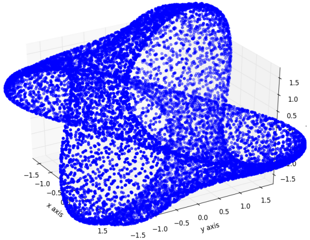

Klein component

Two irreducible components. The surface is connected!

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- Complex:

$$\mathcal{C}(\varepsilon, E) = \bigcup_{p \in E} H(r(p), \varepsilon)$$

  where $H(q, \varepsilon)$ is the hypercube in $\mathbb{R}^8$ centered at q with side length ε.

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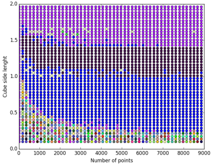

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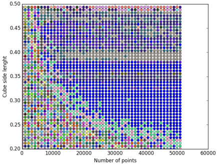

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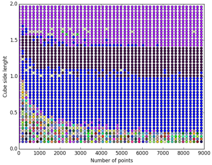

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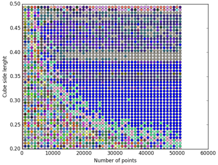

Klein component

Blue corresponding to the homology of a sphere in the case of the spherical component ($H_0 = \mathbb{Z}$, $H_1 = 0$, $H_2 = \mathbb{Z}$) and that of a Klein bottle in the other case ($H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_2 = 0$).

Summary

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- Algebraic Models
  - Kinematics
  - Intersection theory
- Presenting (visualising) a solution
  - Numerical methods
  - Sampling
- Combining the two approaches: Cycloalkane