

Lecture 1 - 2

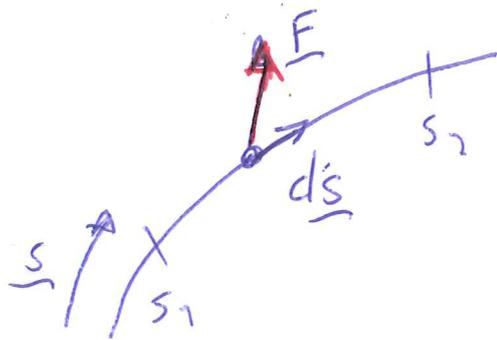
Elastic energy and energy methods

In a loaded elastic body, the work is stored as elastic energy that can be used as useful work

(Think of the spring in a watch...)

Definition of mechanical work

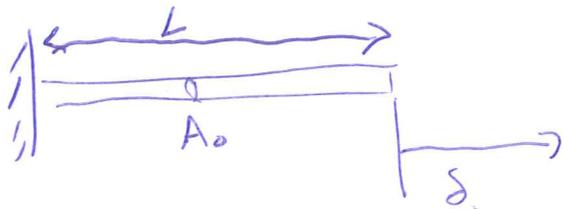
consider a particle moving from s_1 to s_2 and a force \underline{F} is acting on that particle



The work W from the force is then

$$W = \int_{s_1}^{s_2} \underline{F} \cdot \underline{ds}$$

A uniaxial rod:



$$W = \int_0^{\delta} F(\delta) d\delta$$

$$F = \sigma A_0$$

$$\delta = L \epsilon$$

$$d\delta = L d\epsilon$$

$$W = \int_0^{\epsilon} \sigma A_0 L d\epsilon = V \int_0^{\epsilon} \sigma(\epsilon) d\epsilon = V W'$$

W' work per volume / elastic energy / volume

For a linear elastic material

$$\sigma = E \epsilon$$

$$W' = \int_0^{\epsilon} \sigma(\epsilon) d\epsilon = \int_0^{\epsilon} E \epsilon d\epsilon = \frac{E \epsilon^2}{2} = \frac{\sigma^2}{2E}$$

Generalization to 3D

$$W' = \int \sigma_{ij} d\epsilon_{ij} = \int (\sigma_{xx} d\epsilon_{xx} + \sigma_{xy} d\epsilon_{xy} + \dots + \sigma_{zz} d\epsilon_{zz})$$

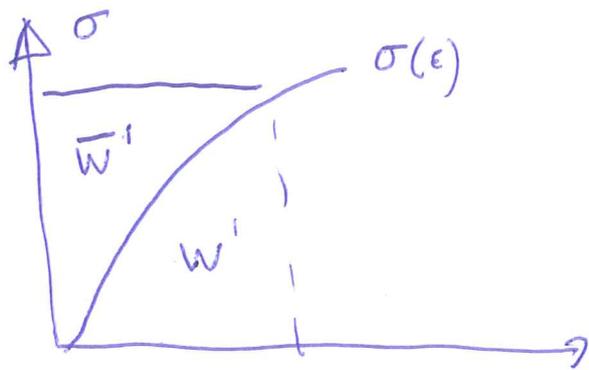
Linear elastic material

$$W' = \frac{1}{2} \sigma_{ij} \epsilon_{ij} = \frac{1}{2} (\sigma_{xx} \epsilon_{xx} + \sigma_{xy} \epsilon_{xy} + \sigma_{zz} \epsilon_{zz})$$

Complementary elastic energy \bar{W}

uniaxial: $\bar{W}' = \int \epsilon(\sigma) d\sigma$ $\epsilon = \frac{d\bar{W}'}{d\sigma}$

3D: $\bar{W}' = \int \epsilon_{ij}(\sigma_{ij}) d\sigma_{ij} = \int (\epsilon_{xx} d\sigma_{xx} + \dots)$

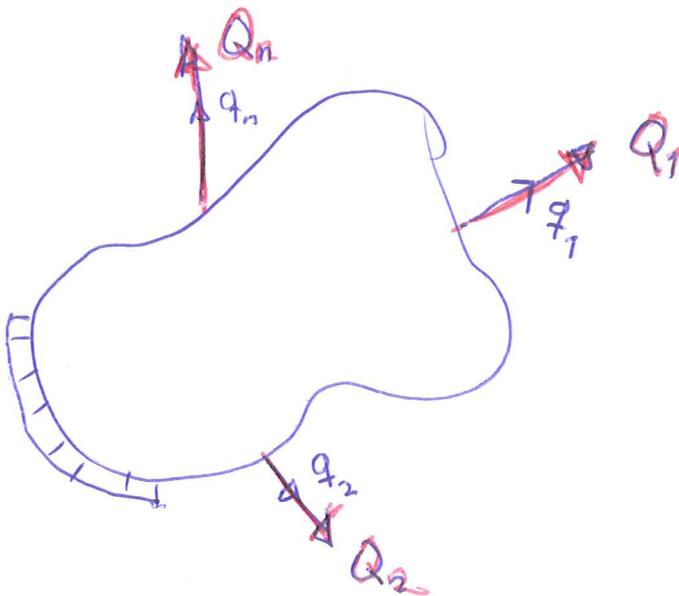


$$W' + \bar{W}' = \sigma \epsilon$$

for a linear elastic material

$$\bar{W}' = W = \frac{1}{2} \sigma \epsilon$$

Structures with discrete degrees of freedom



Q_i - forces, torques
 q_i - displacements, rotations in the direction of Q_i

Elastic system - Independent of the order of application of the forces

$W = W(q_1, q_2, \dots, q_n)$
or (unique relationship between displacement and forces)

$$W = W(Q_1, Q_2, \dots, Q_n)$$

(same for complementary elastic energy)

$$W = \int_0^{q_1} Q_1(q_1) dq_1 + \int_0^{q_2} Q_2(q_2) dq_2 + \dots + \int_0^{q_n} Q_n(q_n) dq_n$$

$$\bar{W} = \int_0^{Q_1} q_1(Q_1) dQ_1 + \dots$$

Castigliano's theorems

1st $Q_k = \frac{\partial W}{\partial q_k}$ known displacements
forces sought

2nd $q_k = \frac{\partial \bar{W}}{\partial Q_k}$ known forces
displacements sought
(HW 1)

Linear elastic system:

Possible to write forces as

$$Q_i = \sum_{j=1}^n k_{ij} q_j \quad \text{or} \quad \underline{Q} = \underline{k} \underline{q} \quad \underline{k} - \text{stiffness matrix}$$

and displacements

$$q_i = \sum_{j=1}^n \alpha_{ij} Q_j \quad \text{or} \quad \underline{q} = \underline{\alpha} \underline{Q} \quad \underline{\alpha} - \text{flexibility matrix}$$

then

$$W(q_1, q_2, \dots) = \frac{1}{2} (q_1 Q_1 + q_2 Q_2 + \dots) = \sum_{i=1}^n \frac{1}{2} q_i Q_i =$$

$$= \sum_{i=1}^n \sum_{j=1}^n q_i k_{ij} q_j = \frac{1}{2} \underline{q}^T \underline{k} \underline{q}$$

and

$$\bar{W}(Q_1, Q_2, \dots) = \frac{1}{2} \underline{Q}^T \underline{\alpha} \underline{Q}$$

$$\frac{\partial W}{\partial q_i \partial q_j} = \frac{\partial}{\partial q_i} \left[\frac{\partial W}{\partial q_j} \right] = \frac{\partial}{\partial q_i} [Q_j] = \frac{\partial}{\partial q_i} \left[\sum_{k=1}^n k_{ik} q_k \right] = k_{ji}$$

$$\frac{\partial W}{\partial q_i \partial q_j} = \frac{\partial}{\partial q_j} \left[\frac{\partial W}{\partial q_i} \right] = \frac{\partial}{\partial q_j} Q_i = \dots = k_{ij}$$

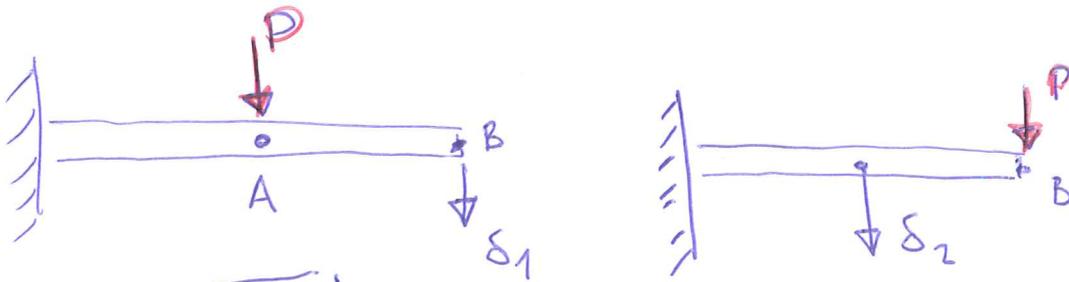
Maxwell's
reciprocal
theorems!

Independent of order of derivation

$$k_{ij} = k_{ji} \quad \text{or} \quad \underline{k} = \underline{k}^T$$

\underline{k} symmetric
& as well

Physical interpretation of the reciprocal theorem

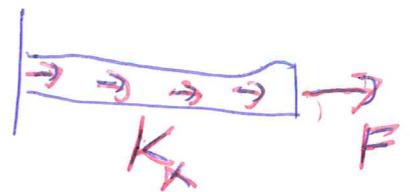


$$\boxed{\delta_1 = \delta_2}$$

Elastic energy for some common elements

A rod with constant cross-section

$$W = \int_L w' A dx = \int_L \frac{1}{2} E \epsilon A dx$$



$$\bar{W} = \int_L \frac{\sigma}{2E} A dx$$

if $k_x = 0$, i.e. $N(x) = \text{konst}$

$$W = \frac{EA}{2L} \delta^2 = \boxed{\frac{1}{2} k \delta^2}$$

$$\bar{W} = \frac{N^2 L}{2EA} = \boxed{\frac{1}{2} \frac{N^2}{k}}$$

k - "spring stiffness"

Beam loaded in pure bending

strain: $\epsilon_x(z) = k z$

\uparrow
curvature

stress: $\sigma_x(z) = \frac{M}{I} \cdot z$

$$W = \int_L \int_A \frac{1}{2} E (k z)^2 dA dx = \int_L k^2 \underbrace{\int_A z^2 dA}_{I} dx =$$

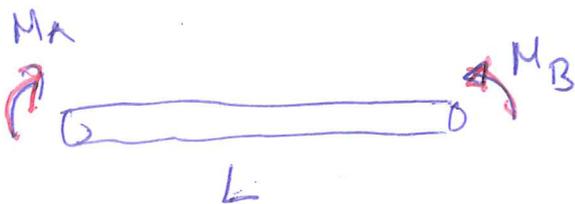
$$= \int_L \frac{1}{2} EI (w'')^2 dx$$

19.2 in Handbook
or 6.51

$$\bar{W} = \int_L \int_A \frac{1}{2} \left(\frac{M}{I} \cdot z \right)^2 \frac{1}{E} dA dx = \int_L \frac{1}{2EI^2} M^2 \int_A z^2 dA dx =$$

$$= \int_L \frac{M^2}{2EI} dx \quad 6.52$$

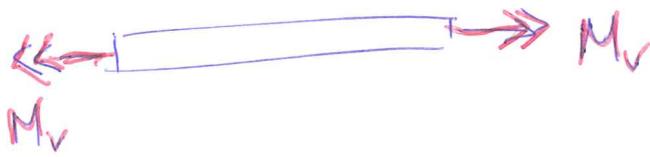
Beam with linearly varying bending moment



$$\bar{W} = \frac{L}{6EI} \left[M_A^2 + M_A M_B + M_B^2 \right]$$

6.53

Shaft loaded in torsion

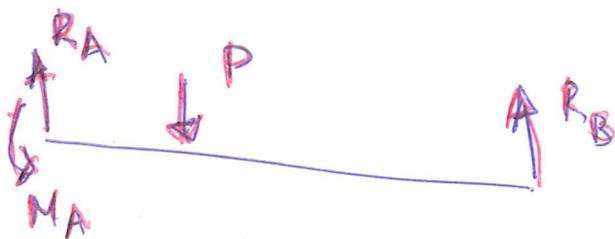
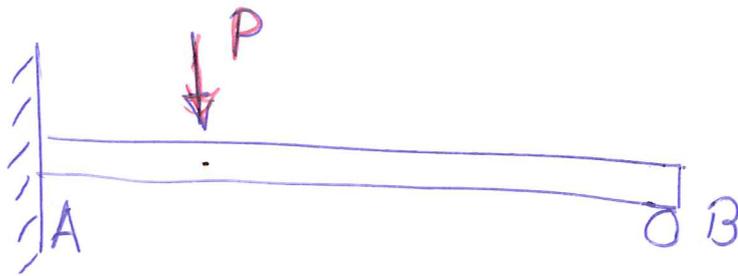


θ - twisting angle

$$W = \frac{1}{2} k_{\theta} \theta^2 = \frac{1}{2} \frac{GK}{L} \theta^2$$

$$\bar{W} = \frac{1}{2} \frac{M_v^2}{k} = \frac{1}{2} \frac{M_v^2 L}{GK}$$

Statically indeterminate structures



3 unknowns

Equilibrium:

$$\uparrow \sum F: R_A + R_B - P = 0$$

$$\curvearrowleft \sum M: -M_A + PL - R_B 3L = 0$$

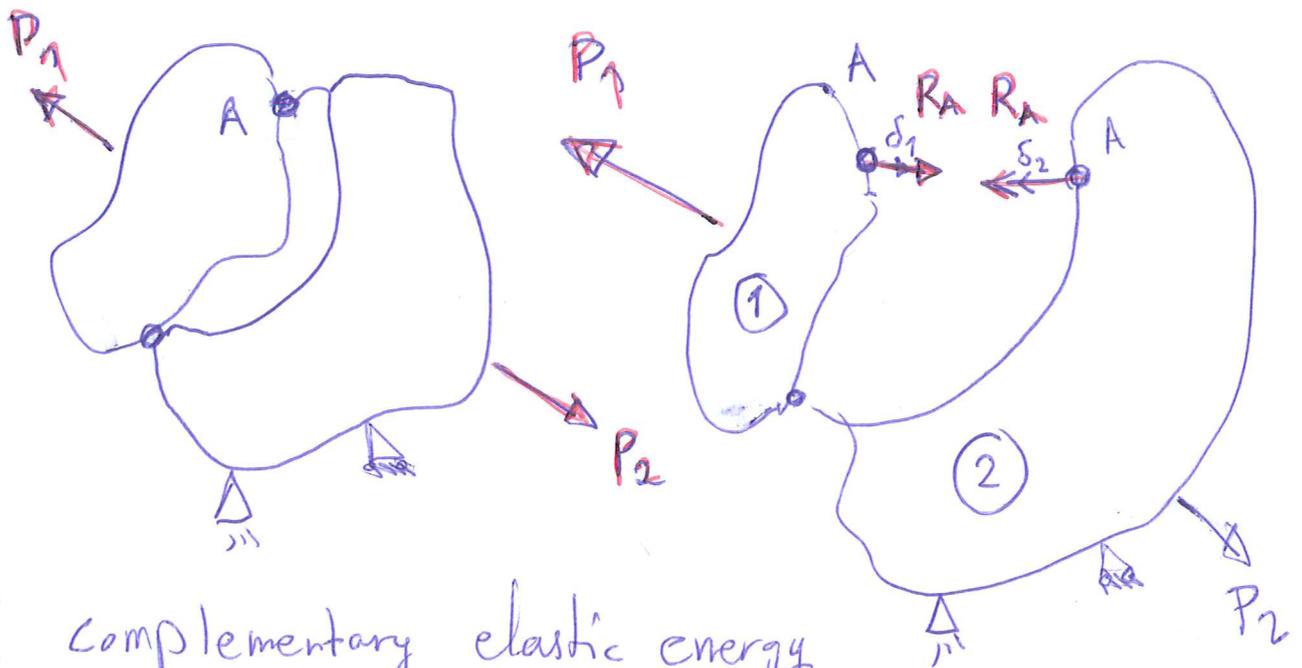
2 Eq.

Solution no vertical displacement at B
 Castigliano's 2nd theorem

$$\frac{\partial \bar{W}}{\partial R_B} = \delta_B = 0 \quad \text{1 extra equation}$$

$$\text{3 eqs 3 unknowns!}$$

Generalization to arbitrary inner force in a structure. Principle of Least Work



complementary elastic energy
 for the substructures, \bar{W}_1, \bar{W}_2

δ_1, δ_2 becomes

$$\delta_1 = \frac{\partial \bar{W}_1}{\partial R_A} \quad \delta_2 = \frac{\partial \bar{W}_2}{\partial R_2}$$

Due to compatibility $\delta_1 = -\delta_2$

$$\frac{\partial (\bar{W}_1 + \bar{W}_2)}{\partial R_A} = \delta_1 + (-\delta_1) = 0$$

or

$$\frac{\partial \bar{W}}{\partial R_A} = 0$$

Summary:

$$\frac{\partial \bar{W}}{\partial P} = \delta$$

and

$$\frac{\partial \bar{W}}{\partial R} = 0$$

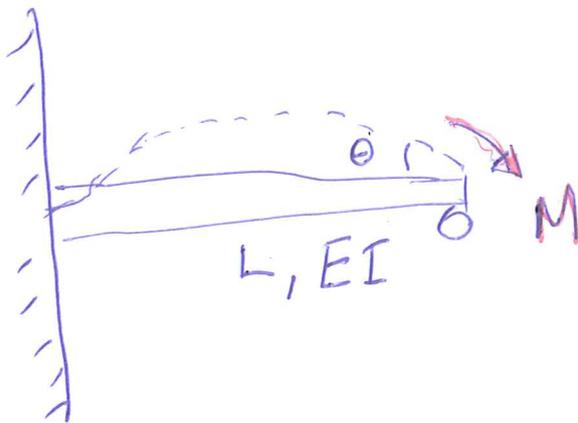
P: External force

R: Internal force

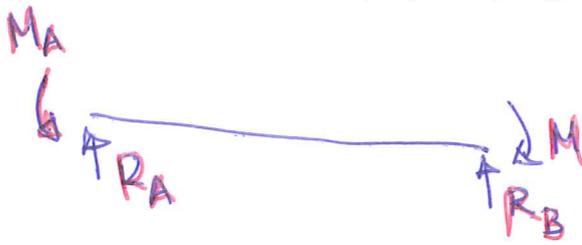
statically indeterminate

δ : displacement in
the direction of P

Example:



Determine θ



$$\sum \bar{M}_A: M - M_A - R_B L = 0$$

$$M_A = M - R_B L$$

Beam with linearly varying bending moment

$$\bar{W} = \frac{L}{6EI} \left[M_A^2 + M_A M_B + M_B^2 \right] =$$

$$= \frac{L}{6EI} \left[M^2 - 2MR_B L + R_B^2 L^2 + M^2 - R_B L M + M^2 \right] =$$

$$= \frac{L}{6EI} \left[3M^2 - 3M R_B L + R_B^2 L^2 \right]$$

Principle of least work

$$\frac{\partial \bar{W}}{\partial R_B} = 0 \Rightarrow -3ML + 2R_B L^2 = 0$$

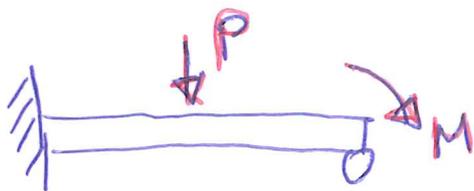
$$R_B = \frac{3M}{2L}$$

$$\bar{W} = \frac{L}{6EI} \left[3M^2 - 3M \frac{3M}{2L} L + \frac{9M^2}{4L^2} L^2 \right] =$$

$$= \frac{L}{6EI} \left[3M^2 - \frac{9M^2}{2} + \frac{9M^2}{4} \right] = \frac{3L}{24EI} M^2$$

$$\theta = \frac{\partial \bar{W}}{\partial M} = \frac{6}{24} \frac{L}{EI} M = \boxed{\frac{M_0 L}{4EI}}$$

For the deflection in the middle of the beam
solve



and set P to zero
in the final answer