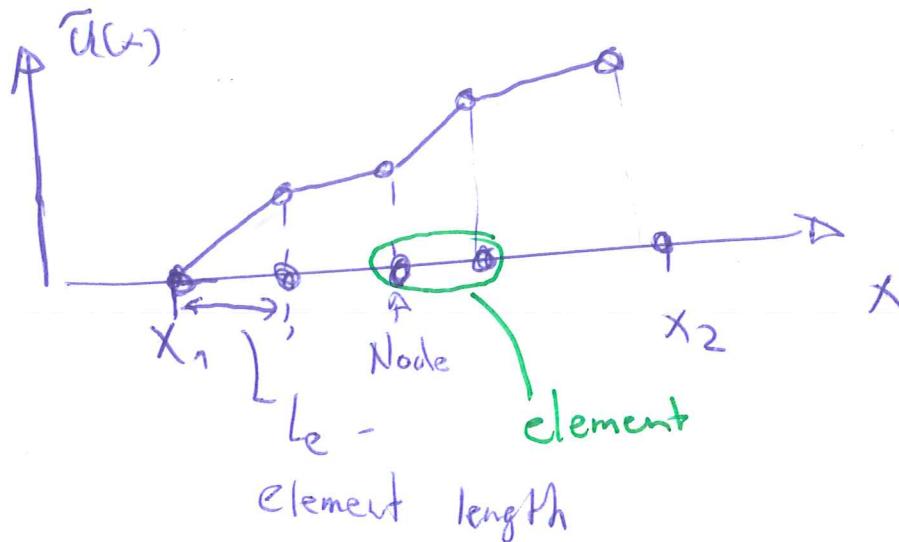


FEM - Approximate solutions of the weak form

Approximate $u(x)$ and $v(x)$ by piecewise continuous functions. Approximation of u , $\tilde{u}(x)$

Example



$\tilde{u}(x)$ and $v(x)$ must satisfy the conditions

- Continuity across element boundaries
- Completeness, i.e. the functions and their derivatives up to the highest order appearing in the weak form must be able to assume constant values

Necessary for convergence
ie $\tilde{u}(x) \rightarrow u(x)$ when $l_e \rightarrow 0$

example: $\tilde{u}(x) = c_0 + c_1 x$ $\tilde{u}'(x) = c_1$ ok

$$\tilde{u}(x) = c_0 + c_2 x^2 \quad \tilde{u}'(x) = 2c_2 x \quad \text{not ok}$$

$$\tilde{u}(x) = c_0 + c_1 x + c_2 x^2 \quad \tilde{u}'(x) = C_1 + 2C_2 x \quad \text{ok}$$

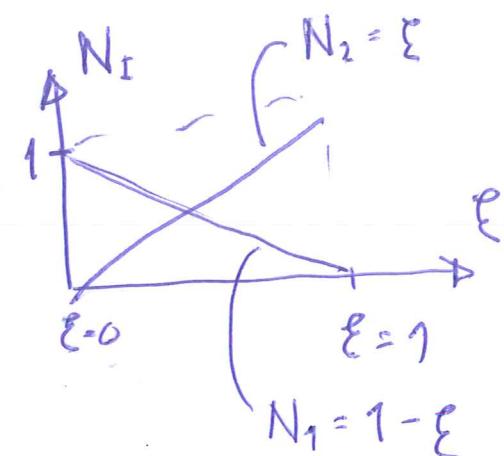
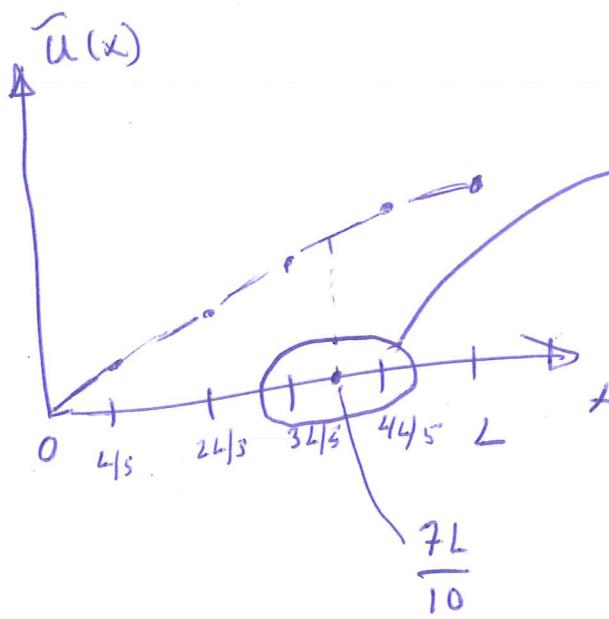
Approximate solution $\tilde{u}(x)$

Formulated by use of shape functions

N_I and node values u_I

N_I - often a polynomial function of a non dimensional coordinate in an element

Example, linear shape functions



Coordinate transformation

$$x = \frac{3L}{5} + \frac{L}{5}\xi$$

$$\begin{aligned} u(\xi) &= N_1(\xi)u_1 + N_2(\xi)u_2 = \\ &= [N_1 \ N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underline{N} \underline{u}_{de} \end{aligned}$$

$$u\left(x = \frac{7L}{10}\right) = u(0.5)$$

Using shape functions of order $n-1$ requires n nodes in the elements. (1 node/coefficient)

$$\tilde{u}(\xi) = \sum_{I=1}^n N_I u_I = \underline{N} \underline{u}_{de}$$

Properties of shape functions

- $N_I(\xi_J) = \begin{cases} 1 & \text{if } J=I \\ 0 & \text{if } J \neq I \end{cases}$

\uparrow
coordinate of node J

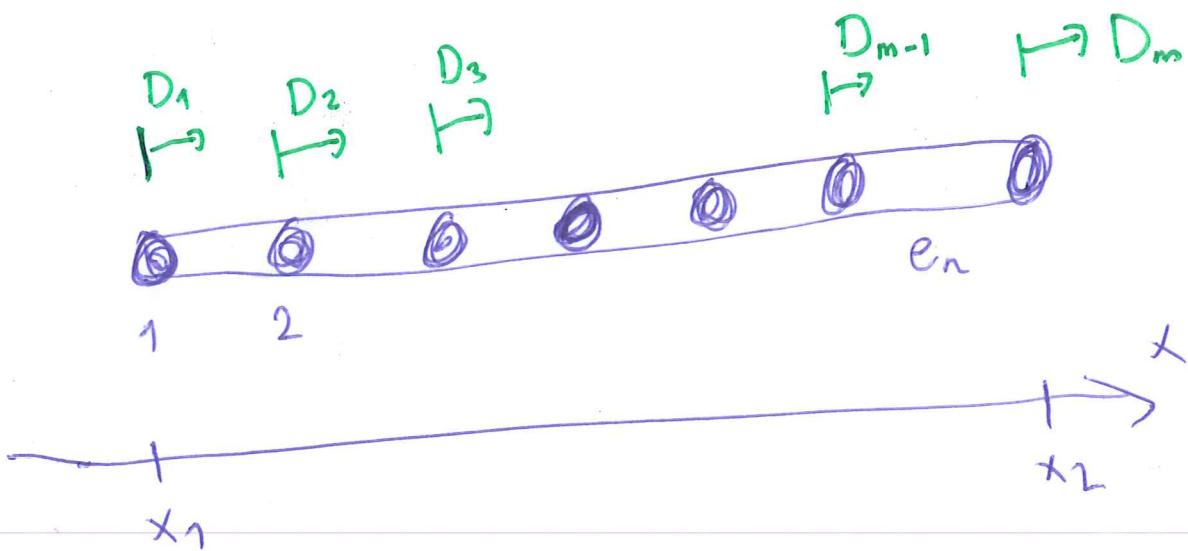
- $\sum_{I=1}^n N_I(\xi) = 1$ do not apply to problems with rotational DOFs (beams, plates...)

The weight function $V(x)$

Use the Galerkin method: same interpolation

as for $\hat{u}(x)$ i.e. $V(\xi) = \sum_{I=1}^n N_I \beta_I = \underline{N} \underline{\beta}_e$

Assembly of all n elements \Rightarrow FEM eqs



Ansatz function and weight function

$$\bar{u} = \underline{\underline{N}}_G \underline{D}$$

$$\bar{r} = \underline{\underline{N}}_G \underline{b} = \underline{\underline{M}}^T \underline{\underline{N}}_G^T$$

$\underline{\underline{N}}_G$ - vector with shape functions for all elements

$$\underline{D} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{m-1} \\ D_m \end{bmatrix} \quad \text{same for } \underline{b}$$

The derivatives!

$$\frac{du}{dx} = \frac{d\underline{\underline{N}}_G}{dx} \underline{D} = \underline{\underline{B}}_G \underline{D}$$

$$\frac{dv}{dx} = \frac{d\underline{\underline{N}}_G}{dx} \underline{b} = \underline{\underline{B}}_G \underline{b} = \underline{\underline{M}}^T \underline{\underline{B}}_G^T$$

Inserted in the weak form

$$\int_{x_1}^{x_2} \frac{dv}{dx} EA \frac{du}{dx} dx - \left(\left[v(x) \underline{\underline{N}} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} v(x) K_x A dx \right) = 0$$

$$\int_{x_1}^{x_2} \underline{B}_G^T \underline{B}_G^T E A \underline{B}_G \underline{D} dx = \left(\underline{B}_G^T \underline{N}_G^T \underline{N}_G \underline{B}_G \right)_{x_1}^{x_2} + \int_{x_1}^{x_2} \underline{B}_G^T \underline{N}_G^T K_x A dx = 0$$

or

$$\underline{B}^T \left[\int_{x_1}^{x_2} \underline{B}_G^T E A \underline{B}_G dx \right] \underline{D} - \left(\left[\underline{N}_G^T \underline{N} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \underline{N}_G^T K_x A dx \right) = 0$$

$\underbrace{\quad}_{K}$

\underline{B} - Vector with arbitrary coefficients, thus \underline{F}

$$\underline{K} \underline{D} - \underline{F} = \underline{0} \quad \text{or} \quad \boxed{\underline{K} \underline{D} = \underline{F}}$$

\underline{K} - Global stiffness matrix

\underline{F} - External load vector

The integrations are simplified as the shape functions are zero outside

$$\begin{aligned} \underline{K} &= \int_{x_1}^{x_2} \underline{B}_G^T E A \underline{B}_G dx = \int_{x_1}^{x_{e1}} \underline{B}_G^T E A \underline{B}_G dx + \int_{x_{e1}}^{x_{e2}} \underline{B}_G^T E A \underline{B}_G dx = \\ &= \sum_{e=1}^n \underline{B}_G^T E A \underline{B}_G dx = \sum_{e=1}^n K_e \quad (\text{assembly}) \end{aligned}$$

same as spring networks)

Example linear elements and

$$B = \frac{dN}{dx} = \frac{dN}{d\epsilon} \frac{\partial \epsilon}{dx}$$

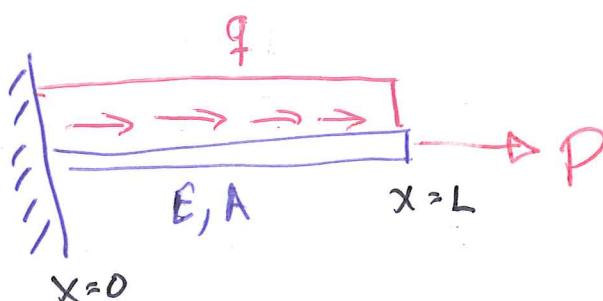
$$x = x_0 + l_e \epsilon$$

$$N = [1-\epsilon, \epsilon] \quad B = [-1, 1] / l_e \quad \text{Point forces on nodes}$$

$$F = \sum_{e=1}^n \left[\int N^T K(\epsilon) A l_e d\epsilon \right]_e + F_s$$

$\underbrace{\qquad\qquad\qquad}_{F_b}$

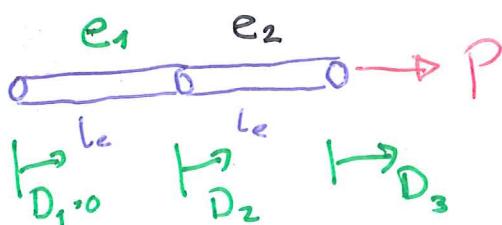
Example



$$w(x) = \frac{PLx}{EA L} + \frac{q_0 L^2}{E} \left(\frac{x}{L} - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right)$$

$$\sigma(x) = \frac{P}{A} + q_0 L \left(1 - \frac{x}{L} \right)$$

FEM solution with two linear elements



Element stiffness matrices

$$K_e = \int_0^{l_e} B^T E A B l_e = \left\{ B = \frac{1}{l_e} [-1, 1] \right\} =$$

$$= \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} F_e &= \int_0^1 N^T K_x(\epsilon) A l_e d\epsilon = \\ &= \int_0^1 \begin{bmatrix} 1-\epsilon \\ \epsilon \end{bmatrix} q A \frac{L}{2} d\epsilon = \\ &= \frac{q AL}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Global stiffness matrix

$$K = \frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

External load vector

$$F_b = \frac{qAL}{4} \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} = \frac{qAL}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad F = \frac{qAL}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} R_1 \\ 0 \\ P \end{bmatrix}$$

Equation system

$$\frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} D_1 = 0 \\ D_2 \\ D_3 \end{bmatrix} = \frac{qAL}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} R_1 \\ 0 \\ P \end{bmatrix}$$

Reduced system, remove row 1 and column 1

$$\frac{2EA}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} D_2 \\ D_3 \end{bmatrix} = \frac{qAL}{4} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ P \end{bmatrix}$$

$$\begin{bmatrix} D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \frac{PL}{2EA} + \frac{3L^2q}{8E} \\ \frac{PL}{EA} + \frac{L^2q}{2AE} \end{bmatrix}$$

Row 1 gives R_1

$$R_1 + \frac{qAL}{4} = \frac{2EA}{L} [1 \cdot D_1 - 1 \cdot D_2 + 0 \cdot D_3] = -P - \frac{3LAq}{4} \Rightarrow R_1 = -P - LAq$$