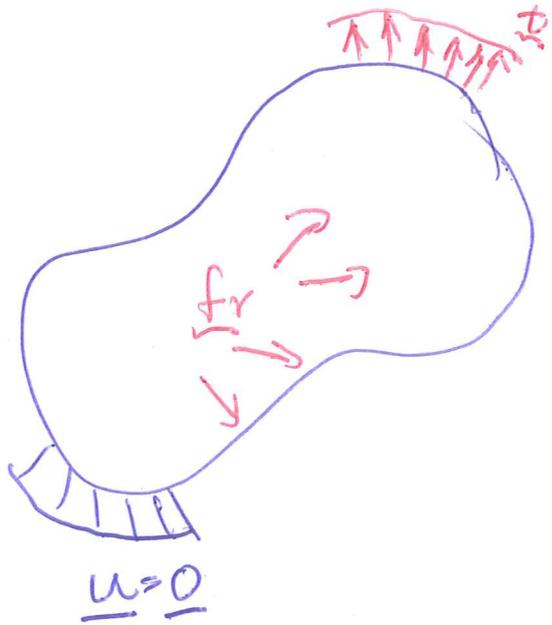


Lecture 11 FEM for 2D/3D solids

3D solids:



Loads:

load on the surface
traction vector \underline{t}

$$\underline{t} = [t_x \ t_y \ t_z]^T$$

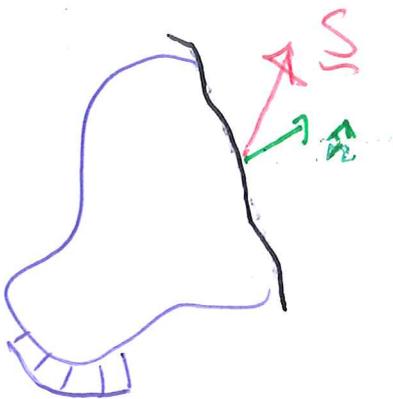
Body force:

$$\underline{f}_v = [K_x \ K_y \ K_z]^T$$

Loads causes displacements

$$\underline{u} = [u_x \ u_y \ u_z]^T \\ = [u, v, w]^T$$

Stresses in 3D



Stress vector \underline{S}

$$\underline{S} = \underline{\sigma} \hat{n}$$

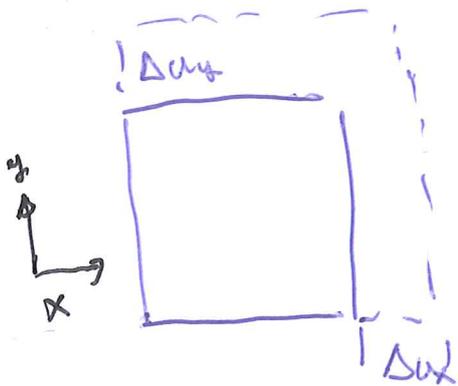
stress tensor $\underline{\sigma}$

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

Vector form components

$$\underline{\sigma} = [\sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \tau_{xy} \ \tau_{xz} \ \tau_{yz}]^T$$

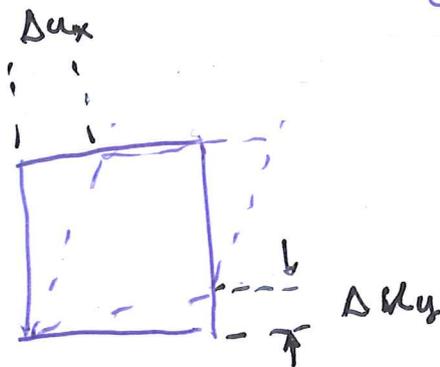
Definition of strains



$$E_{xx} = \frac{\partial u}{\partial x} \quad E_{yy} = \frac{\partial v}{\partial y}$$

$$E_{zz} = \frac{\partial w}{\partial z}$$

Normal strains



Shear strains

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \dots$$

Stored in vector form

$$\underline{\epsilon} = [E_{xx} \ E_{yy} \ E_{zz} \ \gamma_{xy} \ \gamma_{xz} \ \gamma_{yz}]^T$$

$$\underline{\epsilon} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \underline{L} \underline{u}$$

Partial differential operator

Relationship between stresses and strains
constitutive relations.

Elastic material in 1D: $\sigma = E\varepsilon$

Isotropic elastic material in 3D

$$\sigma_{xx} = \frac{E}{1+\nu} \left[\varepsilon_{xx} + \frac{\nu}{1-2\nu} (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \right]$$

$$\sigma_{yy} = \dots$$

$$\sigma_{zz} = \dots$$

$$\tau_{xy} = G \gamma_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}$$

⋮

or in matrix form

$$\underline{\sigma} = \underline{C} \underline{\varepsilon} \quad \underline{\sigma}, \underline{\varepsilon} \text{ 6x1 vectors } \underline{C} \text{ 6x6 matrix}$$

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

$$C_{11} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}$$

$$C_{12} = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

$$\frac{C_{11}-C_{12}}{2} = G$$

Other \underline{C} for orthotropic materials

Elastic energy/unit volume

$$dW' = \sigma_{xx} d\epsilon_{xx} + \sigma_{yy} d\epsilon_{yy} + \sigma_{zz} d\epsilon_{zz} + \tau_{xy} d\gamma_{xy} + \dots = \\ = \underline{d\epsilon^T} \underline{\sigma}$$

Virtual work/unit volume

$$\delta W' = \underline{\delta\epsilon^T} \underline{\sigma}$$

$\underline{\delta\epsilon^T}$ - vector with virtual strain components caused
the virtual displacements $\underline{\delta u} = [\delta u_x, \delta u_y, \delta u_z]^T$

$$\underline{\delta\epsilon} = \underline{L} \underline{\delta u}$$

Principle of virtual work in 3D

$$\delta A^{(i)} = \delta A^{(e)}$$

Internal virtual work:

$$\delta A^{(i)} = \int_V \delta W' dV = \int_V \underline{\delta\epsilon^T} \underline{\sigma} dV$$

External virtual work: Contribution from surface loads + body forces

Integration over the surface of the body

$$\delta A^{(e)} = \int_S \underline{\delta u^T} \underline{t} dS + \int_V \underline{\delta u^T} \underline{f}_v dV$$

$$\delta u_x t_x + \delta u_y t_y + \delta u_z t_z$$

$$\int_V \delta \underline{\underline{\varepsilon}}^T \underline{\underline{\sigma}} dV = \int_S \delta \underline{\underline{u}}^T \underline{\underline{t}} dS + \int_V \delta \underline{\underline{u}}^T \underline{\underline{f}}_v dV$$

$$\delta \underline{\underline{\varepsilon}} = \underline{\underline{L}} \delta \underline{\underline{u}} \Rightarrow \delta \underline{\underline{\varepsilon}}^T = (\delta \underline{\underline{u}} \underline{\underline{L}})^T$$

$$\int_V (\underline{\underline{L}} \delta \underline{\underline{u}})^T \underline{\underline{\sigma}} dV = \int_S \delta \underline{\underline{u}}^T \underline{\underline{t}} dS + \int_V \delta \underline{\underline{u}}^T \underline{\underline{f}}_v dV$$

Linear elastic material

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \underline{\underline{\varepsilon}} = \underline{\underline{C}} (\underline{\underline{L}} \underline{\underline{u}})$$

$$\int_V (\underline{\underline{L}} \delta \underline{\underline{u}})^T \underline{\underline{C}} (\underline{\underline{L}} \underline{\underline{u}}) dV = \int_S \delta \underline{\underline{u}}^T \underline{\underline{t}} dS + \int_V \delta \underline{\underline{u}}^T \underline{\underline{f}}_v dV$$

FEM-solution: Displacement interpolation in 2D/3D

- Divide the solid into N_e elements
- Use "simple" displacement interpolations in each element by use of shape function
- Good to use the same interpolation in all three directions (u_x, u_y, u_z), element with n_d nodes needs then n_d shape functions

$$u(x, y, z) = N_1(x, y, z) u_1 + N_2(x, y, z) u_2 + \dots + N_{nd}(x, y, z) u_{nd}$$

$$v(x, y, z) = N_1(x, y, z) v_1 + \dots$$

$$w(x, y, z) = N_1(x, y, z) w_1 + \dots$$

(u_1, v_1, w_1) - displacement of node 1 in x , y and z -direction

Matrix form:

$$\begin{bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \dots \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix}$$

3x1

3x3nd

or

$$\underline{u} = \underline{N} \underline{d}_e$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{bmatrix}$$

3nd x 1

Virtual displacement field: Use Galerkin's method

$$\underline{\delta u} = \underline{N} \underline{\delta e} \quad \underline{\delta e} \text{ - vector with arbitrary coefficients}$$

$$\underline{\delta u}^T = (\underline{N} \underline{\delta e})^T = \underline{\delta e}^T \underline{N}^T$$

Strains $\underline{\underline{\epsilon}}$

$$\underline{\underline{\epsilon}} = (\underline{\underline{L}} \underline{\underline{u}}) = (\underline{\underline{L}} \underline{\underline{N}}) d\underline{\underline{e}} = \underline{\underline{B}} d\underline{\underline{e}} \quad \underline{\underline{B}} = \underline{\underline{L}} \underline{\underline{N}}$$

virtual strain field

$$\underline{\underline{\delta \epsilon}}^T = (\underline{\underline{L}} \underline{\underline{\delta u}})^T = (\underline{\underline{L}} \underline{\underline{N}} \underline{\underline{B}}_e)^T = (\underline{\underline{B}} \underline{\underline{B}}_e)^T = \underline{\underline{B}}_e^T \underline{\underline{B}}^T$$

Inserted in weak form

$$\int_V \underline{\underline{\delta \epsilon}}^T \underline{\underline{\epsilon}} dV = \int_S \underline{\underline{\delta u}}^T \underline{\underline{t}} dS + \int_V \underline{\underline{\delta u}}^T \underline{\underline{f}}_v dV$$

$$\int_V \underline{\underline{B}}_e^T \underline{\underline{B}}^T \underline{\underline{\epsilon}} dV = \int_S \underline{\underline{B}}_e^T \underline{\underline{N}}^T \underline{\underline{t}} dS + \int_V \underline{\underline{B}}_e^T \underline{\underline{N}}^T \underline{\underline{f}}_v dV$$

$$\underline{\underline{B}}_e^T \left[\int_V \underline{\underline{B}}^T \underline{\underline{\epsilon}} dV \right] - \left(\int_S \underline{\underline{N}}^T \underline{\underline{t}} dS + \int_V \underline{\underline{N}}^T \underline{\underline{f}}_v dV \right) = 0$$

$\underbrace{\hspace{10em}}_{\underline{\underline{K}}_e} \qquad \underbrace{\hspace{10em}}_{\underline{\underline{f}}_e}$

$\underline{\underline{B}}_e^T$ - vector with arbitrary coefficients

$$\underline{\underline{K}}_e \underline{\underline{e}} - \underline{\underline{f}}_e = 0 \quad \text{or} \quad \underline{\underline{K}}_e \underline{\underline{e}} = \underline{\underline{f}}_e$$

For several elements the volume and surface integrations reduces to assembly operation due to the fact that the shape functions are zero outside the elements

The shape functions has their usual properties

$N_i = 1$ in node i , 0 at other nodes

$$\sum N_i = 1$$

Calculations of stresses

$$\underline{\sigma} = \underline{C} \underline{\epsilon} = \underline{C} \underline{B} \underline{d}_e$$

In general \underline{B} will vary with position and thus $\underline{\sigma}$ will not be constant in the elements

For some element types the strains (and stresses) will be constant