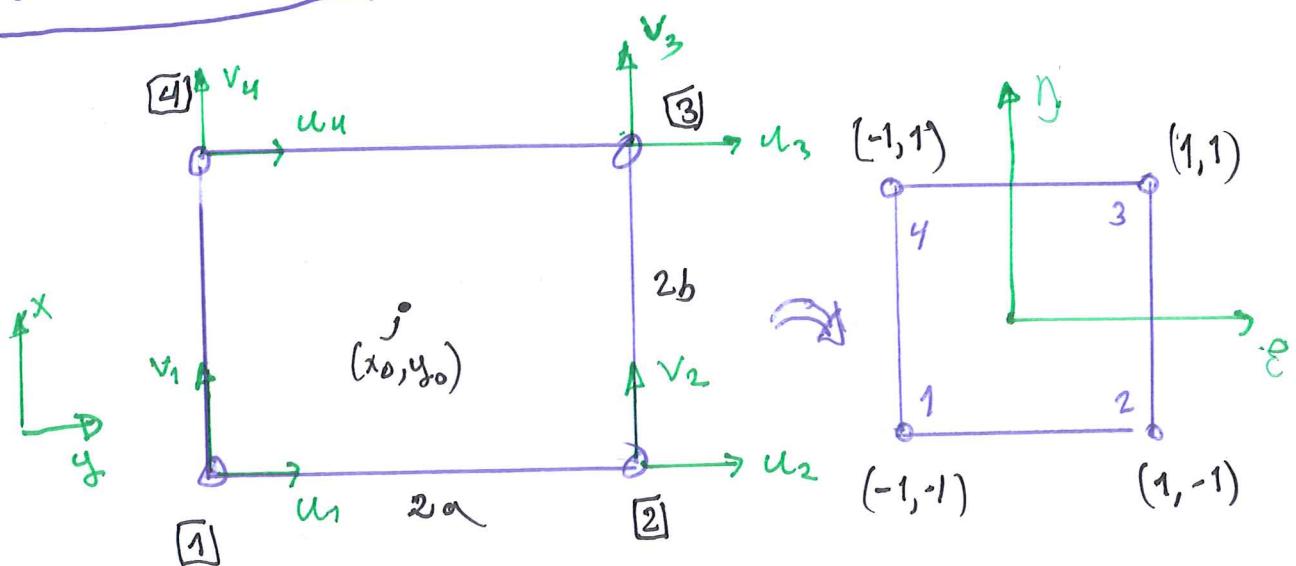


Lecture 13 | 4-noded plane elements



Coordinate transformation

$$\xi = \frac{x - x_0}{a} \quad \eta = \frac{y - y_0}{b}$$

Displacement interpolation

$$\begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & \dots & N_4 & 0 \\ 0 & N_1 & \dots & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$u = \sum N_i u_i$$

Shape functions found by inspection using

$$N_i(\xi_j, \eta_j) = 1 \text{ if } j=i \text{ else } N_i = 0$$

$N_i = 0$ on edges which are not connected to node i

$$\sum_{i=1}^4 N_i = 1$$

The shape functions

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Strains

$$\underline{\epsilon} = \underline{B} \underline{d\epsilon} = \underline{L} \underline{N} \underline{d\epsilon} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \underline{N} \underline{d\epsilon}$$

Partial derivatives of shape functions

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{1}{a} \frac{\partial N_i}{\partial \xi} = \frac{1}{a} N_i, \xi$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{1}{b} \frac{\partial N_i}{\partial \eta} = \frac{1}{b} N_i, \eta$$

$$\underline{B} = \begin{bmatrix} \frac{N_{1,\xi}}{a} & 0 & \frac{N_{4,\xi}}{a} & 0 \\ 0 & \frac{N_{1,\eta}}{b} & 0 & \frac{N_{4,\eta}}{b} \\ \frac{N_{1,\eta}}{b} & \frac{N_{1,\xi}}{a} & \frac{N_{4,\eta}}{b} & \frac{N_{4,\xi}}{a} \end{bmatrix} \quad 8 \times 3 \text{ matrix}$$

Stiffness matrix and external load vector
as usual

$$k_e = h \int_{A_e} B^T C B dA = \left\{ dA = ab d\epsilon dy \right\} = hab \int_{-1}^1 B^T C B d\epsilon dy$$

surface forces

$$f_e = f_p + f_s + f_b$$

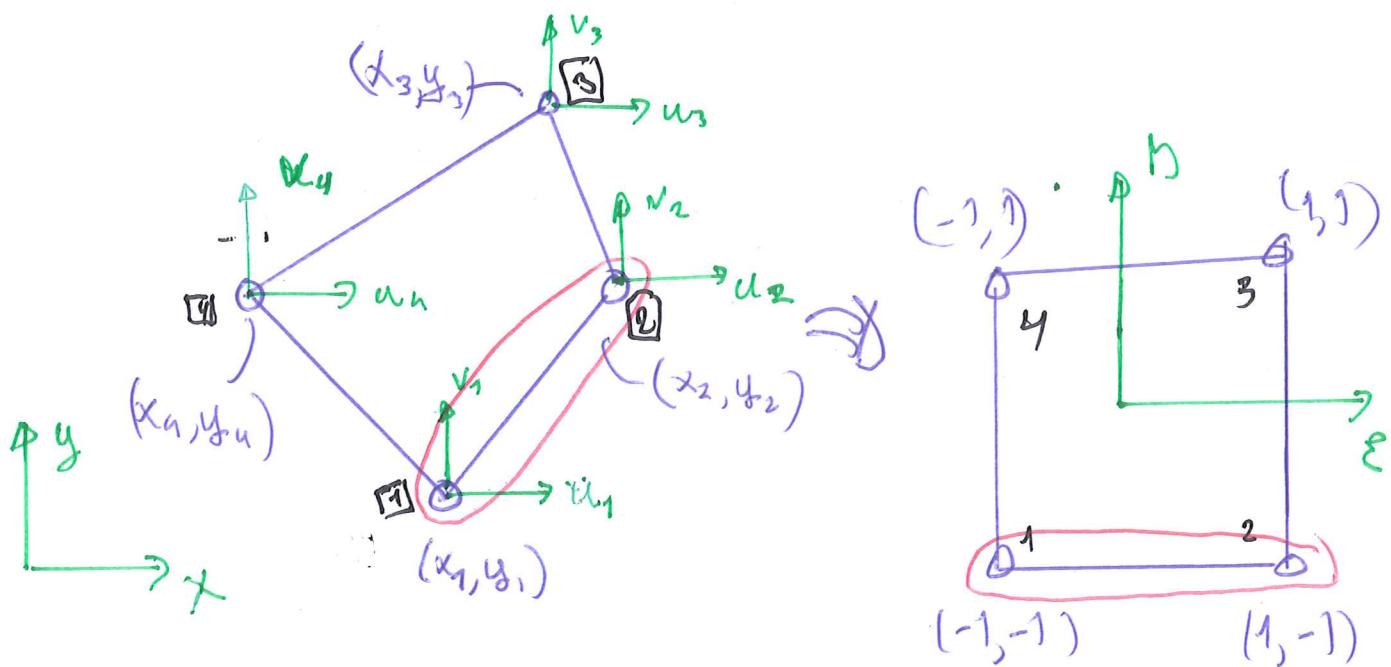
Point forces body forces
on nodes

$$f_s = h \int_L N^T f_s dL$$

le integration around the contour of
the element

$$f_b = h \int_{A_e} N^T f_v dA = hab \int_{-1}^1 N^T f_v d\epsilon dy$$

Bio-parametric elements



Coordinate transformation

Use the shape functions to describe the position in the global coordinate system in the same way as the displacements

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & \dots & N_4 & 0 \\ 0 & N_1 & \dots & 0 & N_4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}$$

or

$$X = N X_e$$

or

$$x(\xi, \eta) \cdot N_1 x_1 + \dots + N_u x_u = \sum_{i=1}^u N_i x_i$$

$$y(\xi, \eta) = N_1 y_1 + \dots + N_u y_u = \sum_{i=1}^u N_i y_i$$

The B-matrix

$$\underline{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \underline{N}$$

Divide into 4 sub-matrices

$$\underline{B} = [\underline{B}_1, \underline{B}_2, \underline{B}_3, \underline{B}_4]$$

$$\underline{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

The partial derivatives $\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}$ are needed

Chain rule for partial derivatives

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta}$$

or

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$$

$\underbrace{\hspace{10em}}$

J

J - Jacobi matrix of the coordinate transformation
using $\underline{x} = \underline{N} \underline{x}_e$

$$J(\xi, \eta) = \begin{bmatrix} \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} \end{bmatrix} \dots \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

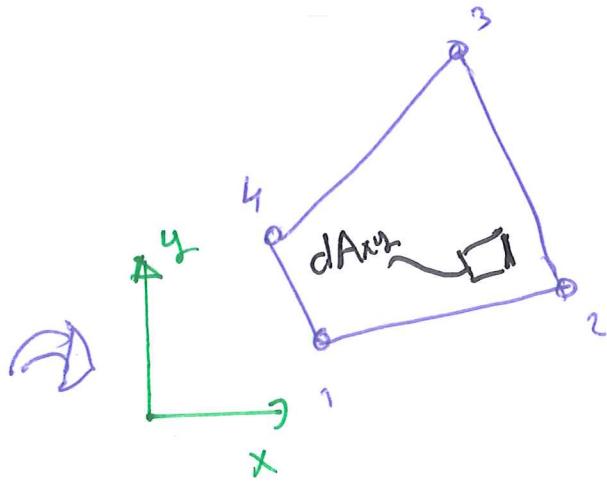
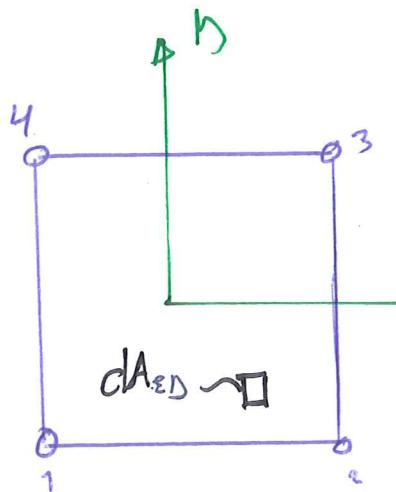
The needed partial derivatives in the B -matrix are then

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

use $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Area relations



$$dA_{xy} = \det(\underline{\underline{J}}) dA_E = \det(\underline{\underline{J}}) d\varepsilon d\gamma$$

Element stiffness matrix

$$\underline{k}_e = h \int_A \underline{\underline{B}}^T \underline{\underline{C}} \underline{\underline{B}} dA = h \int_{-1-1}^{11} \underline{\underline{B}}^T \underline{\underline{C}} \underline{\underline{B}} \det(\underline{\underline{J}}) d\varepsilon d\gamma$$

External force vector

$$\underline{f}_e = \underline{f}_p + \underline{f}_s + \underline{f}_b$$

Point forces
on nodes

$$\begin{aligned} \underline{f}_s &= h \int_e \underline{N}^T \underline{t} dL = h \left[\int_{-1}^1 \underline{N}^T \underline{t} \Big|_{\varepsilon=-1} l_{12} d\varepsilon + \int_{-1}^1 \underline{N}^T \underline{t} \Big|_{\varepsilon=1} l_{23} d\gamma + \right. \\ &\quad \left. + \int_{-1}^1 \underline{N}^T \underline{t} \Big|_{\gamma=1} l_{34} d\gamma + \int_{-1}^1 \underline{N}^T \underline{t} \Big|_{\varepsilon=1} l_{41} d\beta \right] \end{aligned}$$

l_{12}, l_{23} etc are side length divided by 2

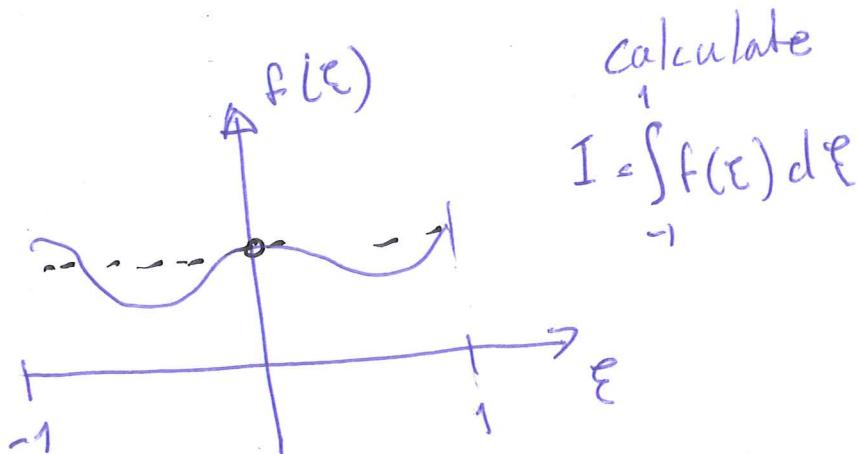
$$l_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\underline{f}_b = h \int_A N^T f_v dA = h \int_{-1}^1 \int_{-1}^1 N^T \begin{bmatrix} K_x \\ K_y \end{bmatrix} \det(J) d\epsilon dy$$

Numeric integration

The integrals over the element area/volume
is hard/impossible to evaluate analytically,
especially for K_e . Thus numeric integration
is needed.

One dimensional case



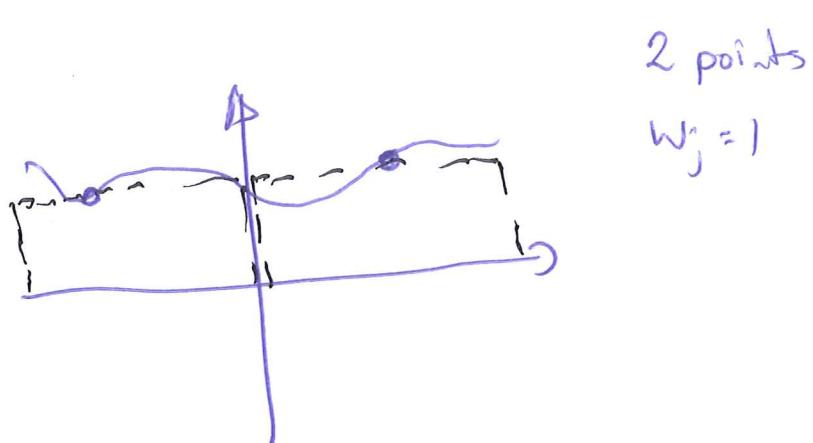
First approximation would be

$$I = 2 \cdot f(0)$$

A general scheme is then

$$I = \sum_{j=1}^m f(\xi_j) w_j$$

evaluate the function at different points and use weights w_j



Gauss integration - evaluate the functions at suitable points, Gauss points, and use corresponding weight coefficients.

It is possible to show that using m points, integration of a polynomial of order $2m-1$ will be exact

More points - higher accuracy but more computational expensive

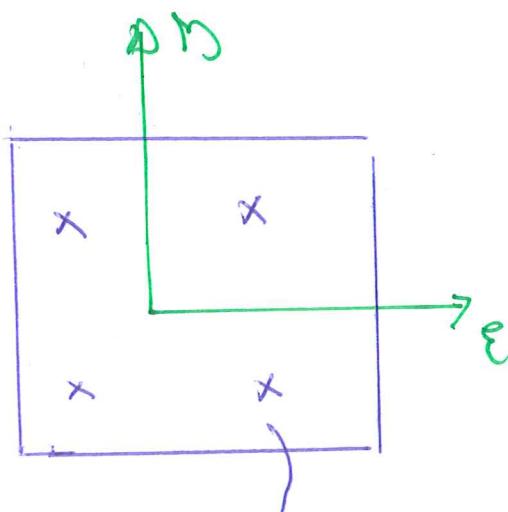
Integration in 2D and 3D

$$k_e = h \int_{A_e} \underline{B}^T \underline{C} \underline{B} dA = h \int_{-1}^1 \int_{-1}^1 \underline{B}^T \underline{C} \underline{B} \det(\underline{J}) d\varepsilon e d\eta$$

$$= h \sum_{i=1}^n \sum_{j=1}^m f(\varepsilon_i, \eta_j) w_i w_j$$

with $f = \underline{B}^T \underline{C} \underline{B} \det(\underline{J})$

In practice 2×2 points are used for linear elements



function evaluated at those points

$$\pm 1/\sqrt{3}, \pm 1/\sqrt{3}$$

Same approach for 3D

2. Numerical integration

1D: $I = \int_L f(x) dx = \int_{-1}^1 f(\xi) |\mathbf{J}| d\xi = \sum_{i=1}^{m_\xi} F(\xi) w_i$

2D: $I = \int_A f(x, y) dA = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) |\mathbf{J}| d\xi d\eta = \sum_{i=1}^{m_\xi} \sum_{j=1}^{m_\eta} F(\xi, \eta) w_i w_j$

3D: $I = \int_V f(x, y, z) dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(\xi, \eta, \zeta) |\mathbf{J}| d\xi d\eta d\zeta =$
 $= \sum_{i=1}^{m_\xi} \sum_{j=1}^{m_\eta} \sum_{k=1}^{m_\zeta} F(\xi, \eta, \zeta) w_i w_j w_k$

Taken from "The finite element method", G.R. Liu & S.S. Quek

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CHAPTER 7 FEM FOR TWO-DIMENSIONAL SOLIDS

Table 7.1. Gauss integration points and weight coefficients

m	ξ_j	w_j	Accuracy n
1	0	2	1
2	$-1/\sqrt{3}, 1/\sqrt{3}$	1,1	3
3	$-\sqrt{0.6}, 0, \sqrt{0.6}$	5/9, 8/9, 5/9	5
4	$-0.861136, -0.339981,$ $0.339981, 0.861136$	0.347855, 0.652145, 0.652145, 0.347855	7
5	$-0.906180, -0.538469, 0,$ $0.538469, 0.906180$	0.236927, 0.478629, 0.568889, 0.478629, 0.236927	9
6	$-0.932470, -0.661209, -0.238619,$ $0.238619, 0.661209, 0.932470$	0.171324, 0.360762, 0.467914, 0.467914, 0.360762, 0.171324	11