

Lecture 16

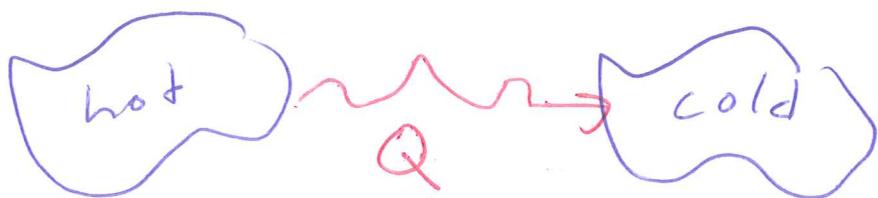
- Heat conduction, governing equations
- FEM for heat conduction problems

Next time

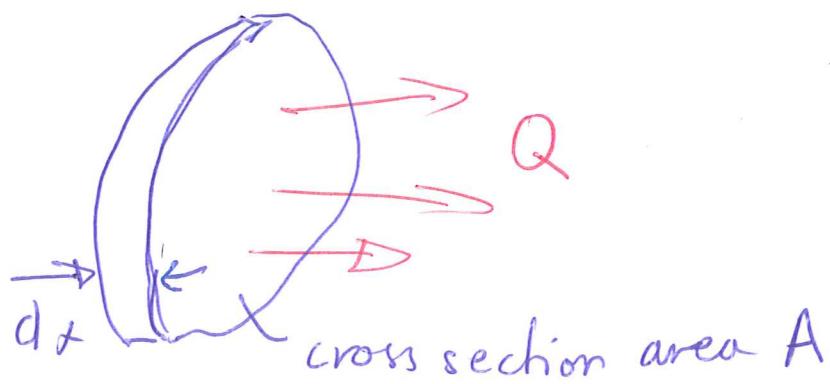
- • FEM for thermo-elastic materials
- • FEM for heat problems in 2D / (3D)

Heat conduction

Heat flows from hot areas to cold



- Heat flow Q , measured in W (power)
- Heat flow through a medium



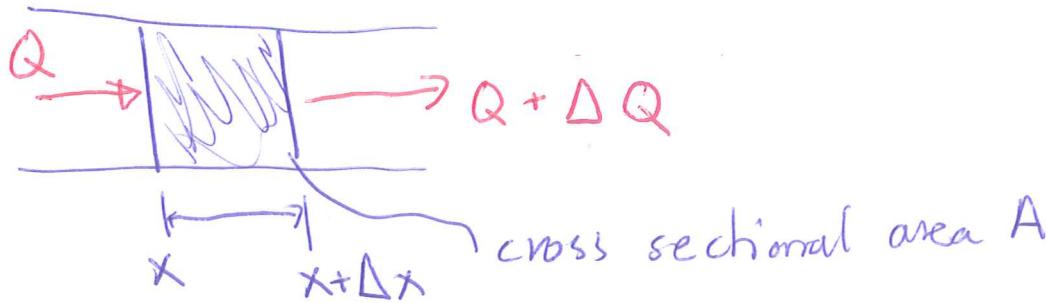
Fouier's law

$$Q = -\frac{dT}{dx} A k$$

k - coefficient of thermal conductivity

unit $\frac{W}{m^{\circ}C}$ or $\frac{W}{mK}$

Power balance for a small mass element



- heat flow in - heat flow out + heat generated
= heat change in the element

or density

$$Q - (Q + \Delta Q) + q A \Delta x = \frac{\partial T}{\partial t} \cdot A \Delta x \rho$$

$A \Delta x \rho$ - mass of the element

q - heat generated/volume [W/m^3] the element

Think of chemical reactions...

c - Specific heat of the material $\left[\frac{\text{J}}{\text{kg}^\circ\text{C}} \right]$

The energy increase of a substance with mass m when increasing the temperature ΔT is

$$\Delta W = cm \Delta T$$

Divide by Δx and let $\Delta x \rightarrow 0$

$$-\frac{\Delta Q}{\Delta x} + q A = \frac{\partial T}{\partial t} c A \rho \quad \text{or} \quad -\frac{\partial Q}{\partial x} + q A = c A \rho \frac{\partial T}{\partial t}$$

Replacing Q with Fourier law $Q = -\frac{\partial T}{\partial x} Ak$

$$\frac{\partial}{\partial x} \left[Ak \frac{\partial T}{\partial x} \right] + qA = cAp \frac{\partial T}{\partial t}$$

DE for heat conduction

In this course we will only consider

steady-state conditions, i.e. $\frac{\partial}{\partial t} = 0$

$$\frac{\partial}{\partial x} \left[Ak \frac{\partial T}{\partial x} \right] + qA = 0$$

$$\text{cf. } \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) + K_x A = 0$$

Typical case: Temperature distributions after long time

Boundary conditions

- Prescribed temperature analogous to prescribed displacement
- Prescribed heat flow $Q = -kA \frac{dT}{dx}$ analogous to prescribed forces
Important special case $Q=0$ implies insulated surface. No heat exchange with the surroundings

- Convection $Q = h(T - T_{\infty}) A$
 h - convection coefficient
 T_{∞} - Temperature of the surrounding medium

- Radiation $Q = \epsilon \sigma (T^4 - T_{\infty}^4) A$
Emission to the surrounding medium
 ϵ - Emissivity of the surface $0 \leq \epsilon \leq 1$

σ - Stefan-Boltzmann constant

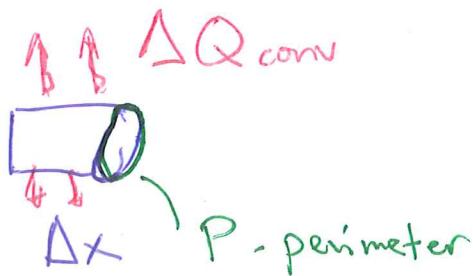
Will not be treated further in the course

Convection

- Convection does not only occur at the boundaries $Q_{\text{convection}}$



Idea: Account for convection
by introducing a
negative heat source



"Convected" power from the element Δx

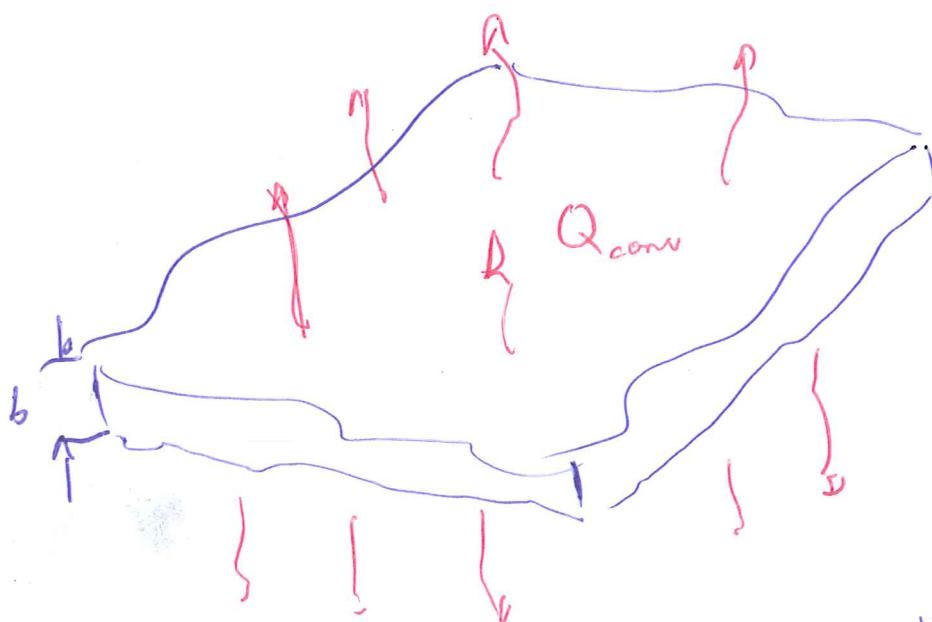
$$\Delta Q_{\text{conv}} = \underbrace{hP}_{\text{surface area}} \Delta x (T - T_{\infty})$$

Assume that this heat is "created" by a volume source q_{conv}

$$q_{\text{conv}} A \Delta x = h P \Delta x (T - T_{\infty})$$

$$q_{\text{conv}} A = h P (T - T_{\infty})$$

In 2D



$$q_{\text{conv}} b \Delta x \Delta y = 2 h \Delta x \Delta y (T - T_{\infty})$$

convection on 2 sides

$$q_{\text{conv}} b = 2 h (T - T_{\infty})$$

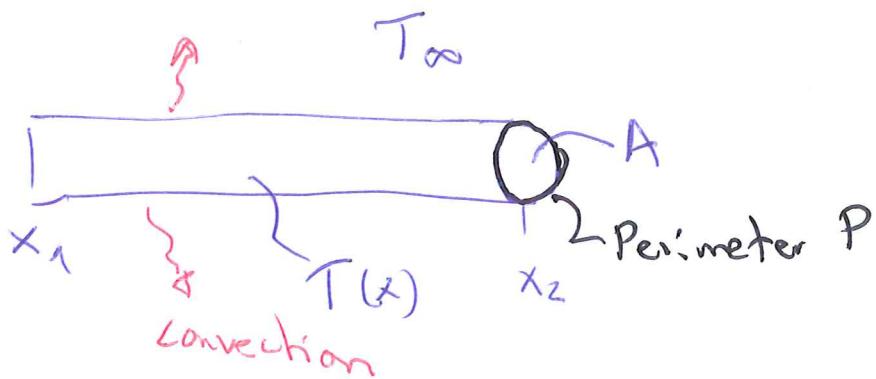
| | |
|---------|--------------------------------|
| Units | |
| 1D: h | $\left[\frac{W}{m} \right]$ |
| 2D: h | $\left[\frac{W}{m^2} \right]$ |

Summary

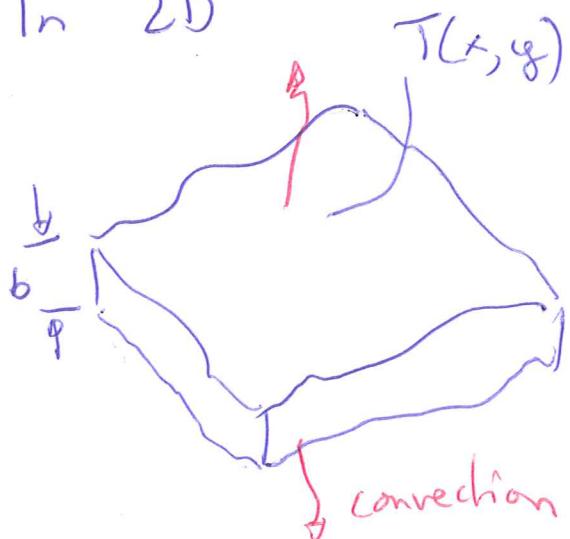
Stationary heat equation 1D

$$\frac{d}{dx} \left(kA \frac{dT}{dx} \right) + qA - hP(T - T_\infty) = 0$$

↑ ↑ ↑
 heat conduction heat supply convection



In 2D



$$\frac{\partial}{\partial x} \left(k_x b \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y b \frac{\partial T}{\partial y} \right) + qb - 2h(T - T_\infty) = 0$$

FEM - equations

Two steps

1. Weak form

2. Interpretation in elements using shape functions

Weak form

Multiply with arbitrary weight function

$v(x)$ and integrate over the problem

$$\int_{x_1}^{x_2} v(x) \left[\frac{d}{dx} \left(kA \frac{dT}{dx} \right) + q_A - hP(T - T_\infty) \right] dx = 0 \quad (*)$$

Integrating the first term by parts

$$\int_{x_1}^{x_2} v(x) \frac{d}{dx} \left(kA \frac{dT}{dx} \right) dx = \left[v(x) kA \frac{dT}{dx} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dv}{dx} kA \frac{dT}{dx} dx$$

- $\int_{x_1}^{x_2} \frac{dv}{dx} kA \frac{dT}{dx} dx$

move the second term and $\int_{x_1}^{x_2} v(x) hP T dx$ to
the other side of (*)

$$Q = -kA \frac{dT}{dx}$$

$$\int_{x_1}^{x_2} \frac{dv}{dx} kA \frac{dT}{dx} dx + \int_{x_1}^{x_2} v LPT dx = \left[-vQ \right]_{x_1}^{x_2} + \\ + \int_{x_1}^{x_2} v q A dx + \int_{x_1}^{x_2} v LPT_\infty dx$$

$T(x)$ only on the left hand side

- - - - - -
- Divide the structure into elements
- In each element, assume

$$T(x) = \underbrace{N}_{\text{matrix}} \underbrace{\underline{T}_e}_{\text{vector containing node temperatures}}$$

Choose weight function according to Galerkin

$$v(x) = \underbrace{N}_{\text{matrix}} \underbrace{\underline{B}}_{\text{vector}} = \underline{B}^T \underline{N}$$

Derivatives

$$\frac{dT}{dx} = \frac{d\underbrace{N}_{\text{matrix}} \underline{T}_e}{dx} = \underline{B} \underline{T}_e$$

$$\frac{dv}{dx} = \frac{d\underbrace{N}_{\text{matrix}} \underline{B}}{dx} = \underline{B}^T \underline{B}$$

Inserted in the weak form

$$\int_{x_1}^{x_2} \underline{\underline{B}}^T \underline{\underline{k}} \underline{\underline{A}} \underline{\underline{B}} \underline{\underline{T}}_e dx + \int_{x_1}^{x_2} \underline{\underline{B}}^T \underline{\underline{N}}^T h \underline{\underline{P}} \underline{\underline{N}} \underline{\underline{T}}_e dx = \\ - \left[\underline{\underline{B}}^T \underline{\underline{N}}^T Q \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \underline{\underline{B}}^T \underline{\underline{N}}^T q A dx + \int_{x_1}^{x_2} \underline{\underline{B}}^T \underline{\underline{N}}^T h P T_\infty dx$$

$\underline{\underline{B}}^T$ = vector with arbitrary coefficients
Thus

$$\int_{x_1}^{x_2} \underline{\underline{B}}^T k \underline{\underline{A}} \underline{\underline{B}} dx + \left[\underline{\underline{N}}^T h \underline{\underline{P}} \underline{\underline{N}} dx \right] \underline{\underline{T}}_e = \\ - \left[\underline{\underline{N}}^T Q \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \underline{\underline{N}}^T q A dx + \int_{x_1}^{x_2} \underline{\underline{N}}^T h P T_\infty dx$$

or

$$(\underline{\underline{K}}_{hc} + \underline{\underline{K}}_c) \underline{\underline{T}}_e = - \left[\underline{\underline{N}}^T Q \right] + f_b$$

FEM-ehw for heat conduction in 1D