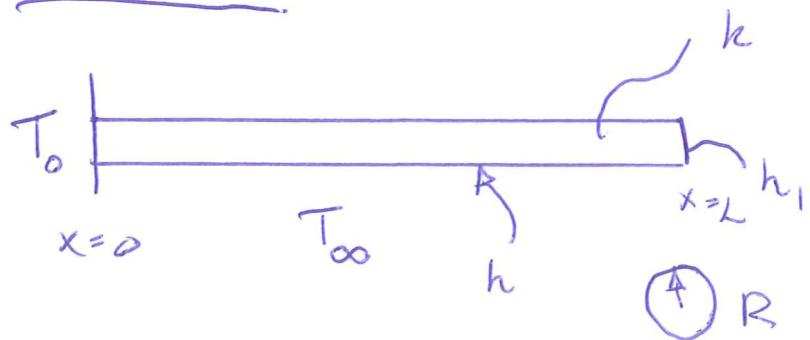
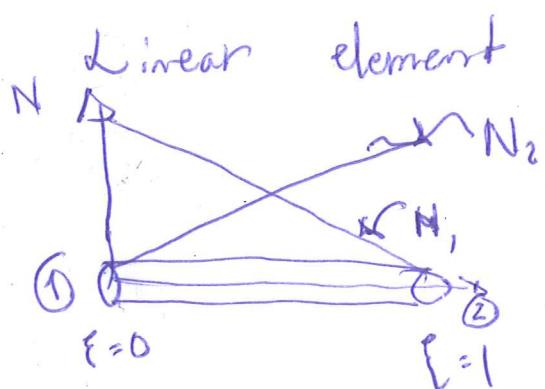


Example



Convection both along the rod and at $x=L$
Different parameters due to different surface properties

Model the rod with 2 first order elements and calculate the temperature distribution



$$\underline{N} = [1 - \xi, \xi]$$

$$x = x_0 + L_e \xi$$

$$dx = L_e d\xi$$

$$\underline{B} = \frac{d\underline{N}}{dx} = \frac{1}{L_e} \frac{d\underline{N}}{d\xi} = \frac{1}{L_e} [-1, 1]$$

$$K_{hc} = \int_{x_0}^{x_1} \underline{B}^T k A \underline{B} dx = \left\{ kA = \text{const} \right\} = kA \int_0^1 \underline{B}^T \underline{B} dx =$$

$$= \frac{kA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{cases} A = \pi R^2 \\ L_e = \pi/2 \end{cases} = \frac{2k\pi R^2}{\pi} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_c = \int_{x_1}^{x_2} N^T h P^- dx = \{hP = \text{const}\} = h P \int_{x_1}^{x_2} N^T N \text{d}x =$$

$$= \frac{hPle}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left\{ \begin{array}{l} P = 2\pi R \\ le = L/2 \end{array} \right\} = \frac{\pi h R l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembly to global system matrix

$$K = \frac{2h\pi R^2}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\pi h R l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2+2 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

$$= \frac{2h\pi R^2}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\pi h R l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The convection B.C. Element 2

$$\begin{bmatrix} N^T(-Q) \end{bmatrix}_{4/2}^2 = \begin{bmatrix} 0 \\ -Q_3 \end{bmatrix} - \begin{bmatrix} -Q_2 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} Q_2 \\ -Q_3 \end{bmatrix} = \begin{bmatrix} Q_2 \\ -hA(T_3 - T_\infty) \end{bmatrix} =$$

$$= - \begin{bmatrix} 0 \\ hAT_3 \end{bmatrix} + \begin{bmatrix} Q_2 \\ hAT_\infty \end{bmatrix} =$$

$$- \begin{bmatrix} 0 & 0 \\ 0 & hA \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} 0 \\ hAT_\infty \end{bmatrix} + \begin{bmatrix} Q_2 \\ 0 \end{bmatrix}$$

Right hand side term due to convection at the surface

For each element

$$\int_{x_1}^{x_2} N^T h P \hat{T}_\infty dx = h P T_{\text{ext}} \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] = \frac{h \pi R T_{\text{ext}} L}{4} \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$

Total assembled vector will then be

$$\frac{\pi h R T_{\text{ext}} L}{2} \left[\begin{array}{c} 1 \\ 2 \\ 1 \end{array} \right]$$

The first term is moved to the left hand side as it contains unknown nodal temperatures. The last term will cancel during assembly due to an opposite heat flow from element 1 in that node

The resulting equation system matrix will then be

$$K = \left[\frac{2k\pi R^2}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\pi h R L}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_1 \pi R^2 \end{bmatrix} \right]$$

And the equation system

$$K \begin{bmatrix} T_1 - T_0 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} Q_1 \\ 0 \\ h_1 A T_\infty \end{bmatrix} + \frac{\pi h R T_{0,0,L}}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Reaction
Prescribed
heat flow due to
temperature

Reduced equation system remove row and column 1

~~$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\pi h R L}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_1 \pi R^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_1 \pi R^2 \end{bmatrix}$$~~

~~$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{\pi h R L}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_1 \pi R^2 \end{bmatrix}$$~~

$$K \begin{bmatrix} T_1 = T_0 \\ T_2 \\ T_3 \end{bmatrix} = F$$

$$K_{11}\bar{T}_1 + K_{12}\bar{T}_2 + K_{13}\bar{T}_3 = F_1$$

$$K_{21}\bar{T}_1 + K_{22}\bar{T}_2 + K_{23}\bar{T}_3 = F_2$$

$$K_{31}\bar{T}_1 + K_{32}\bar{T}_2 + K_{33}\bar{T}_3 = F_3$$

or

$$1 \quad K_{12}\bar{T}_2 + K_{13}\bar{T}_3 = F_1 - K_{11}\bar{T}_1$$

$$2 \quad K_{22}\bar{T}_2 + K_{23}\bar{T}_3 = F_2 - K_{12}\bar{T}_1$$

$$3 \quad K_{32}\bar{T}_2 + K_{33}\bar{T}_3 = F_3 - K_{13}\bar{T}_1$$

2 and 3

$$\underline{K^{\text{red}}}\bar{T}_{\text{red}} = \underline{F^{\text{red}}} - \begin{bmatrix} K_{12} \\ K_{13} \end{bmatrix} \bar{T}_1$$

$$\bar{T}_{\text{red}} = \begin{bmatrix} \bar{T}_2 \\ \bar{T}_3 \end{bmatrix}$$

Another option is to set $\bar{T}_1 = 0$ and
set $T_{\infty} = T_{\infty} - T_0$

Solution gives T_2 and T_3 ($\text{Ug}/\text{y}!$)

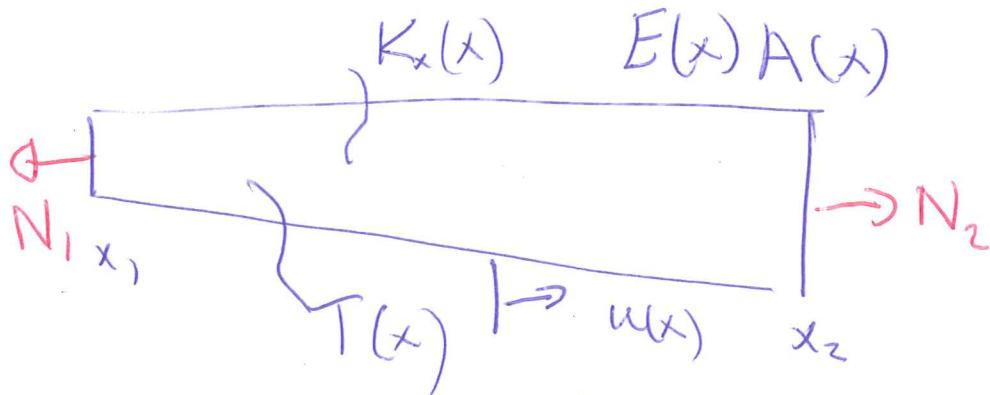
$T(x)$ for $x < L/2$

$$T(\xi) = N \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad \xi = \frac{2x}{L}$$

$T(x)$ for $L/2 < x < L$

$$T(\xi) = N \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} \quad \xi = \frac{2(x - \frac{L}{2})}{L}$$

FEM for Thermo-Elastic materials 1D



Equilibrium: $\frac{dN}{dx} + K_x A = 0$

or $\frac{d(\sigma A)}{dx} + K_x A = 0$

Holds independent of material

Inserted in weak form

$$\int_{x_1}^{x_2} \frac{dv}{dx} \sigma A dx = [v(\sigma A)]_{x_1}^{x_2} + \int_{x_1}^{x_2} v k_x A dx$$

Constitutive relation

$$\epsilon = \frac{\sigma}{E} + \alpha \Delta T$$

temperature change from reference temp

α - Thermal expansion coefficient order $10^{-5} [1/K]$

or

$$\sigma = E \epsilon - E \alpha \Delta T$$

Inserted in the weak form $\epsilon = \frac{du}{dx}$

$$\int_{x_1}^{x_2} \frac{dv}{dx} EA \frac{du}{dx} dx - [v(\sigma A)]_{x_1}^{x_2} + \int_{x_1}^{x_2} v K_x A dx +$$

$$+ \int_{x_1}^{x_2} \frac{dv}{dx} EA \alpha \Delta T dx$$

FEM-cqs for one element $u = \underline{N} \underline{d}_e$

$$v = \underline{B^T} \underline{N}^T$$

$$\Delta T = \underline{N} \underline{\Delta T}_e$$

$\underline{\Delta T}_e$ - change in nodal temperatures

$$\left[\int_{x_1}^{x_2} \underline{B^T} \underline{E} \underline{A} \underline{B} dx \right] \underline{d}_e = [\underline{N}^T \sigma A]_{x_1}^{x_2} + \int_{x_1}^{x_2} \underline{N}^T K_x A dx +$$

$$+ \int_{x_1}^{x_2} \underline{B^T} \underline{E} \underline{A} \underline{N} \underline{\Delta T}_e dx$$

$$\text{or } \underline{k_e} \underline{d}_e = \underline{f}_S + \underline{f}_D + \underline{f}_T$$

Post-processing \rightarrow stress calculation

$$\sigma = E \epsilon - E \alpha \Delta T$$

$$\sigma = E \underline{B}^e \underline{\epsilon}^e - E \alpha \Delta T$$

use the same interpolation for the ΔT
as for the strain

$$\sigma = E \underline{B}^e \underline{\epsilon}^e - E \alpha \widetilde{\Delta T}$$

$\widetilde{\Delta T}$ The mean temperature change in the
element

$$\frac{\Delta T_1 + \Delta T_2}{2}$$

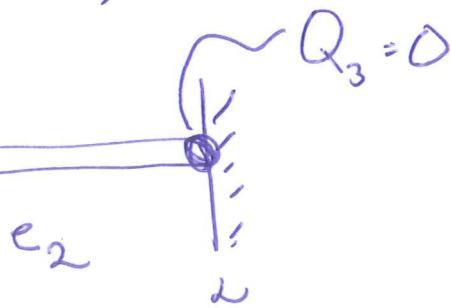
Example

$$T_1 = T_0$$

$x=0$

$$L = 1 \text{ m}$$

EA, $k \propto$



$$E = 200 \text{ GPa}$$

$$A = 0.4 \text{ cm}^2$$

$$P = 2.8 \text{ cm}^2$$

$$\alpha = 14 \cdot 10^{-6}$$

$$k = 3 \text{ W/cm/}^\circ\text{C}$$

$$h = 0.1 \text{ W/cm}^2/\text{C}$$

$$T_0 = 80^\circ \quad T_\infty = 20^\circ$$

convection from the surface, h perimeter P

Start by solving the temperature problem

$$\left(\frac{2kA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{PhcL}{12} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right) \begin{bmatrix} T_1 = T_0 \\ T_2 \\ T_3 \end{bmatrix} =$$

$$\frac{hPT_\infty L}{4} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$K_{hc} = \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_{ec} = \frac{Phc}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$f_c = \int_0^L N^T h P T_\infty dx$$

$$\begin{bmatrix} T_2 \\ T_3 \end{bmatrix} \approx \begin{bmatrix} 25 \\ 21 \end{bmatrix}^\circ\text{C}$$

Thermo-elastic analysis. Assume that

The rod had $T(x) = 20^\circ$ from the beginning

$$\frac{2EA}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} D_1 = 0 \\ D_2 \\ D_3 = 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ R_3 \end{bmatrix} + f_T$$

f_T for 1 element

$$\int_{x_1}^{x_2} \underline{B^T} EA\alpha \underline{N} \Delta T_e dx = \int_0^1 EA\alpha e \underline{B^T} \underline{N} \Delta T_e dx =$$

$$= EA\alpha \int_0^1 -[(1-\varepsilon)\Delta T_1 + \varepsilon \Delta T_2] d\varepsilon =$$

$$= EA\alpha \left[\frac{-\Delta T_1 + \Delta T_2}{2} \right]$$

$$f_T = \frac{EA\alpha}{2} \begin{bmatrix} -(\Delta T_1 + \Delta T_2) \\ (\Delta T_1 + \Delta T_2) - (\Delta T_2 + \Delta T_3) \\ \Delta T_2 + \Delta T_3 \end{bmatrix}$$

$$K_{red} D_{red} = F_{red} \Rightarrow D_2 = F_2 | K_{22}$$

$$D_2 = 8.26 \mu m$$

$$\sigma = E B \frac{de}{dx} - E \alpha \overline{\Delta T}$$

Element 1

$$\sigma^{(1)} = E \frac{\alpha}{2} [-1, 1] \begin{bmatrix} 0 \\ D_2 \end{bmatrix} - E \alpha \frac{\Delta T_1 + \Delta T_2}{2} = \\ = -19.8 \text{ MPa}$$

$$\sigma^{(2)} = \dots = -19.8 \text{ MPa}$$

Makes sense that $\sigma^{(1)} = \sigma^{(2)}$