

# CHAPTER 4

## Two-Dimensional Multirate Filter Banks for Subband Coding

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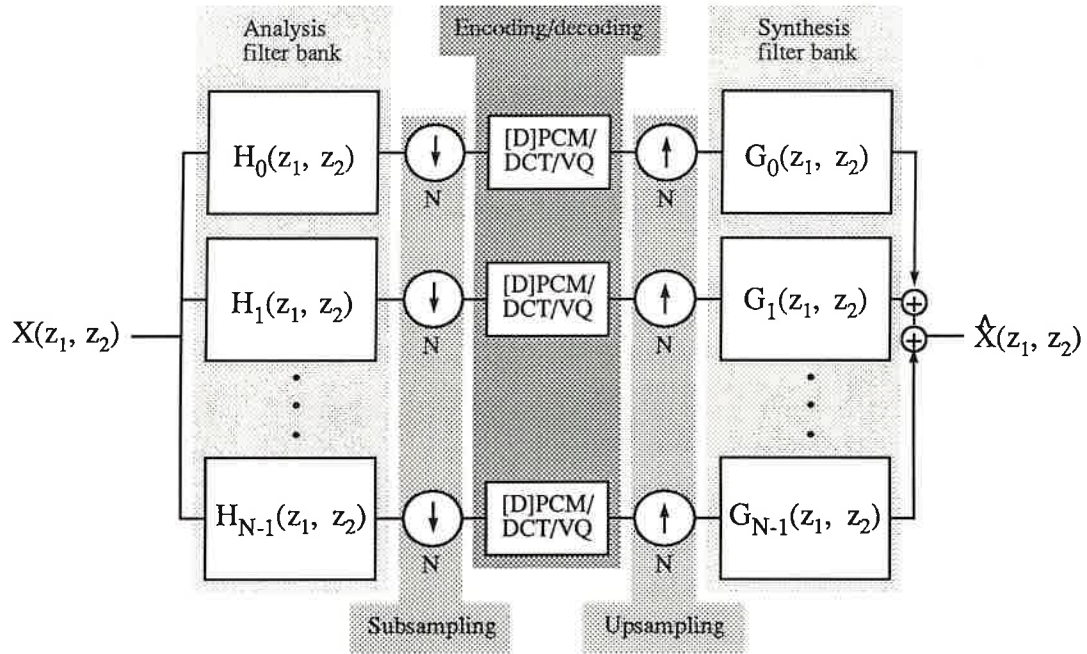
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In recent years, subband coding has gained attention as a powerful method for compressing still images and video (see for example [VET84, VBR86a, VBR86b, WOO86, GHA86, GHA87, SMI87b, WES87, KAR87, GHA88, KAR88a, LEG88, WES88a, KAR88b, ANS88]). The technique of subband coding is explained in Fig. 4.1. A signal is passed through a bank of spatial bandpass filters, the analysis filters. Owing to the reduced bandwidth, each resulting component may be subsampled to its new Nyquist frequency, thus yielding the subband signals. Following that, each subband would be encoded, transmitted, and, at the destination, decoded. To finally reconstruct the signal, each subband is upsampled to the sampling rate of

the input. All upsampled components are passed through the synthesis filter bank, where they are interpolated, and are added to form the reconstructed signal.

Recall that a discrete signal has the spectrum repeated at points which are multiples of the sampling frequency. The subsampling moves these spectra together so that they will partly overlap. Thus, the low frequency part of one spectrum will be shifted into the high frequency part of another and the signal will be distorted. This effect is referred to as aliasing. It also means that the subsampling of the subbands make the overall transfer function of the system periodically shift variant. The impulse response of the overall system from  $x$  to  $\hat{x}$  will therefore depend on where on the input lattice the impulse is applied. So, aliasing is clearly undesirable. Subband analysis and synthesis should therefore aim at a level of quality which gives alias-free signal reconstruction. This ensures that the system is shift-invariant. In the absence of coding and transmission loss, it is also possible to eliminate any spectral, amplitude and phase distortion that the two filter banks may cause. That is referred to as perfect reconstruction and it will be the level of quality we aim for in this thesis.

The method of subband coding appears to compare favorably to other methods such as transform coding, vector quantization, and predictive coding with regard to compression and quality, computational complexity, and possibility for parallel implementation. Transform and vector-quantization methods are commonly applied on sub-blocks of the image, typically of size  $8 \times 8$ , or  $16 \times 16$  pixels for transform coding, or  $4 \times 4$  pixels for vector quantization. When the block of transform coefficients or pixels is coarsely quantized, the sub-block structure may become visible in the decoded image. This is often referred to as “blocking effects.” Moreover, with the block processing, redundancy due to correlation across block



*Figure 4.1 A subband coding system consists of filter banks, sample-rate conversions and encoding. The latter is commonly performed by PCM, DPCM, DCT, vector quantization, or any combination thereof.*

boundaries is not removed. In subband coding, the images are processed in their entirety and therefore the aforementioned problems do not exist. Subband coding also allows the compression to be adjusted according to perceptual criteria, which may reduce the visibility of the introduced distortion. This relies on the masking properties of the human visual system and the compression is made highest in those spatial frequency bands where the distortion becomes least visible. (This is also possible with Fourier and discrete-cosine transform coding.)

All previously reported work on subband coding of images has used separable processing where the signal is processed as a one-dimensional signal in one direction at a time. Filter bank theory for one-dimensional systems have been thoroughly researched. For an overview of recent results, which are pertinent to the theory of this chapter, the reader is referred to [VET86, SMI86, SMI87a, VET87, VAI87a,

DOG88, VET89] and references therein. It is only proper that a two-dimensional signal, such as an image, should be processed with a two-dimensional system. The advantage of such filter banks, when used in subband coding of images, is that they may have directional properties which are not limited to the vertical and horizontal directions of separable filters, and they enable the use of general, non-rectangular, subsampling patterns. In addition, non-separable two-dimensional filters can have better frequency characteristics than their separable counterpart: consider that a non-separable impulse response of  $M \times M$  coefficients has  $M^2$  free variables while the separable one has only  $2M$ . Multi-directional subband analysis can thereby be obtained which may give enhanced coding performance. Discussions on two-dimensional filter banks, related to the present work, have appeared in [VAI87b, ADE87, SIM88, VIS88, KAR88c, ANS89].

Subband coding systems may be viewed as consisting of three distinct parts (see Fig. 4.1) where the first part is the filter banks which are used for analysis and synthesis of a signal, the second part is the sample rate conversion of the subbands (*i.e.*, sub- and upsampling), and the third part is the encoding and decoding of the subbands. In this chapter the concern is the former two parts; implementation and coding results are presented in Chapters 5 and 6. The theory pertains mainly to perfect reconstruction through finite impulse-response systems and there are no restrictions on the sampling lattices. Furthermore, it is generally applicable to two-dimensional linear systems and it may be restricted to apply to one-dimensional systems as well.

For clarification, we provide examples throughout the text, all of which are based on the useful case of hexagonal subsampling. We denote matrices by boldface capital letters, column vectors by an over-bar (*e.g.*,  $\bar{v}$ ), functions in the space-domain are

named by small letters and their  $z$ -transform \* equivalent are given in capital letters. Thus, the impulse response of the  $i$ th analysis filter is denoted by  $h_i(n_1, n_2)$  and its corresponding transfer function in the  $z$ -transform domain is  $H_i(z_1, z_2)$  (see Fig. 4.1). Analogously, the synthesis filters will be named  $g_i(n_1, n_2)$  and  $G_i(z_1, z_2)$  in the respective domains. The exceptions to this notation are ‘ $N$ ’, which denote the subsampling factor, and ‘ $W_N$ ’ which by convention denotes the  $N$ th root of unity. The axes of the input lattice are  $n_1$  and  $n_2$ , and the axes on the subsampling lattice are  $u_1$  and  $u_2$ . Functions in the polyphase domain are indicated by subscript ‘p’, ‘T’ denotes matrix transpose, and all other notations are explained in the text or given by context.

#### 4.1 Two-dimensional subsampling

An important part of subband coding is the subsampling of the analyzed signal and the reciprocal upsampling before the synthesis. Since the signal is two-dimensional, the subsampling should not be restricted to be along the axes of the input lattice. Let us point out initially that we will only consider sampling structures of the input and the subsampling which are representable as lattices and structures which can only be represented as a union of shifted lattices will not be included. So, assume an image signal which is defined on a general sampling lattice (not limited to the rectangular one). A subsampling of this signal can be seen as a linear transformation from the input lattice to a subsampling lattice [MER83, DUD84, DUB85]. Let this transformation be defined by the subsampling matrix

$$\mathbf{D} = \begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{pmatrix}, \quad (4.1)$$

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\* The two-dimensional  $z$ -transform of a discrete-space function,  $x(n_1, n_2)$ , is defined as  $X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$ .

with integer elements  $d_{ij}$ . We will require the matrix  $\mathbf{D}$  to have  $d_{10} = 0$ ,  $d_{00} > 0$ ,  $d_{11} > 0$ , and  $0 \leq d_{01} < d_{00}$ . This does not lead to loss of generality since a given sub-lattice can be described by more than one  $\mathbf{D}$ -matrix, as shown in Ex. 1. These matrices are all related to one another by rotation matrices with unity determinants [MER83, DUD84, DUB85]. Hence, we have just chosen a form that will simplify our analysis.

Given a location  $(u_1, u_2)$  on the subsampling lattice, the corresponding location  $(n_1, n_2)$  on the sampling lattice of the input is given by

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} d_{00} & d_{01} \\ 0 & d_{11} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (4.2)$$

The requirement that  $d_{10} = 0$  makes the axes  $n_1$  and  $u_1$  collinear. The subsampling factor is given by the determinant of  $\mathbf{D}$ , and, owing to the aforementioned restrictions, it is equal to

$$N = \det (\mathbf{D}) = d_{00} d_{11}. \quad (4.3)$$

For a system with critical subsampling, where the number of samples is conserved, the number of subbands is equal to the subsampling factor [VET87]. The effective subsampling factors along the directions  $n_1$  and  $n_2$  (*i.e.*, the distance between two samples along those axes) can be found to be

$$N_{n_1} = d_{00}, \quad N_{n_2} = c \frac{d_{00}d_{11}}{d_{01}} \quad \text{for } d_{01} > 0, \quad (4.4a)$$

where  $c$  is the smallest integer such that  $N_{n_2}$  is integer, and

$$N_{n_1} = d_{00}, \quad N_{n_2} = d_{11} \quad \text{for } d_{01} = 0. \quad (4.4b)$$

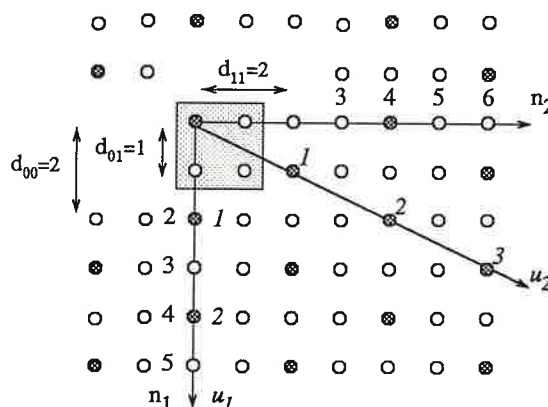
Note that  $N \neq N_{n_1}N_{n_2}$  except when the subsampling is separable, as in Eq. (4.4b).

*Example 1* Hexagonal subsampling can be described by either  $\mathbf{D}_1$  or  $\mathbf{D}_2$  below, of which we opt for the form  $\mathbf{D}_2$ . This subsampling is also illustrated in Fig. 4.2. The

two matrices are related to one another by a matrix with determinant equal to 1.

$$\mathbf{D}_1 = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{D}_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{D}_2 = \mathbf{D}_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$\mathbf{D}_2$  gives subsampling by factors 2 and 4 in the directions of  $n_1$  and  $n_2$ , respectively.



**Figure 4.2** Hexagonal subsampling on a rectangular lattice. The circles denote samples of the input of which the shaded ones are retained in the subsampling. The shaded rectangle is a unit cell—a polyphase component will be defined for each location in this cell (see Section 4.2).

Assume that a signal  $x(n_1, n_2)$  is subsampled, as described by  $\mathbf{D}$  in Eq. (4.1) with the aforementioned restrictions, and immediately upsampled by  $\mathbf{D}^{-1}$ . The resulting signal,  $y(n_1, n_2)$ , can be expressed as the input signal modulated by a function  $f(n_1, n_2)$  (which gives the shift-variance), i.e.,

$$y(n_1, n_2) = f(n_1, n_2) x(n_1, n_2). \quad (4.5)$$

The modulation function is defined on the input lattice,  $I$ . It has to be unity-valued at the points of the subsampling lattice,  $S$ , and zero at all other points of  $I$ :

$$f(n_1, n_2) = \begin{cases} 1 & (n_1, n_2) \in S \\ 0 & (n_1, n_2) \in \bar{S} \quad (= I \setminus S) \end{cases} \quad S \subseteq I. \quad (4.6)$$

It is easily verified that  $f(n_1, n_2)$  can be written as

$$\begin{aligned} f(n_1, n_2) &= \frac{1}{N} \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} \exp \left( -j2\pi (n_1 \ n_2) (\mathbf{D}^{-1})^T \begin{pmatrix} k \\ l \end{pmatrix} \right) \\ &= \frac{1}{N} \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} W_N^{d_{11}n_1k + d_{00}n_2l - d_{01}n_2k}, \end{aligned} \quad (4.7)$$

where  $W_N = e^{-j\frac{2\pi}{N}}$  is the  $N$ th root of unity. If the  $z$ -transforms of  $x(n_1, n_2)$  and  $y(n_1, n_2)$  are  $X(z_1, z_2)$  and  $Y(z_1, z_2)$ , respectively, then, following the modulation theorem, Eq. (4.5) can be expressed as

$$Y(z_1, z_2) = \frac{1}{N} \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} X \left( W_N^{-d_{11}k} z_1, W_N^{-(d_{00}l - d_{01}k)} z_2 \right). \quad (4.8)$$

This expression for the modulation, caused by the sub- and upsampling, will be used to derive the behavior of the entire subband coding system in Section 4.3. We will refer to the term for which  $k = 0$  and  $l = 0$  as the baseband and the terms for which  $k > 0$  and  $l > 0$  as the aliasing-terms. (Alias-free reconstruction means that all aliasing-terms are cancelled so that the output of the system is a filtered version of the baseband.) It can be shown that  $Y(z_1, z_2)$  is a polynomial in powers of  $(z_1^{d_{00}})$  and  $(z_1^{d_{01}} z_2^{d_{11}})$  only.

**Example 2** For the hexagonal subsampling by  $\mathbf{D}_2$  in Ex. 1, Eq. (4.8) yields the modulation

$$\begin{aligned} Y(z_1, z_2) &= \frac{1}{4} (X(z_1, z_2) + X(z_1, -z_2) + X(-z_1, -jz_2) + X(-z_1, jz_2)) \\ &= \Xi(z_1^2, z_1 z_2^2). \end{aligned}$$

## 4.2 Polyphase representation of two-dimensional filter banks

As previously pointed out, the modulation caused by the sub- and upsampling has made the system periodically shift-variant. In fact there will be  $N$  different impulse

responses of the system. In the hexagonal sampling case of Fig. 4.2, there will be different responses for impulses applied at each of the four locations within the shaded unit cell of the input lattice (an impulse at any other location will still only produce one of these four responses). The locations that produce the same impulse response form a coset of the input lattice, such as the shaded circles in Fig. 4.2. The  $z$ -transform of samples which are located on points belonging to the same coset is called a polyphase component. There are thus  $N$  polyphase components which together cover all the points on the input lattice.

Given the subsampling pattern, the analysis filters can be decomposed into polyphase components. Let the filter coefficients of the  $i$ th filter in the analysis filter bank be indexed as  $h_i(n_1, n_2)$  ( $i \in [0, N)$  for critical subsampling), where  $n_1$  and  $n_2$  are the indices along the two axes of the impulse response. Given Eq. (4.2), the  $N$  polyphase components of the  $i$ th filter can be defined as

$$H_{pi,k,l}(z_1, z_2) = \sum_{u_1=-\infty}^{\infty} \sum_{u_2=-\infty}^{\infty} h_i \left( \underbrace{k + d_{00}u_1 + d_{01}u_2}_{n_1}, \underbrace{l + d_{11}u_2}_{n_2} \right) z_1^{-u_1} z_2^{-u_2}, \quad (4.9)$$

where  $k = 0, \dots, d_{00} - 1$  and  $l = 0, \dots, d_{11} - 1$ . So,  $k$  and  $l$  span a unit cell of the input lattice, as illustrated in Fig. 4.2 where the shaded samples will form polyphase component  $H_{pi,0,0}$ . The above definition is a generalization of the familiar one-dimensional polyphase concept [CRO83].

The  $z$ -transform of the impulse response is calculated from the polyphase components by

$$\begin{aligned} H_i(z_1, z_2) &= \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} H_{pi,k,l} \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) z_1^{-k} z_2^{-l} \\ &= \bar{Z}_1(z_1)^T \mathbf{H}_{pi} \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) \bar{Z}_2(z_2). \end{aligned} \quad (4.10)$$

In the second equality, the  $N$  polyphase components of filter  $i$  are given in matrix

form

$$\mathbf{H}_{pi}(z_1, z_2) = \begin{pmatrix} H_{pi,0,0}(z_1, z_2) & \cdots & H_{pi,0,d_{11}-1}(z_1, z_2) \\ \vdots & \ddots & \vdots \\ H_{pi,d_{00}-1,0}(z_1, z_2) & \cdots & H_{pi,d_{00}-1,d_{11}-1}(z_1, z_2) \end{pmatrix}, \quad (4.11)$$

and

$$\bar{Z}_1(z_1) = (1 \quad z_1^{-1} \quad z_1^{-2} \quad \cdots \quad z_1^{-d_{00}+1})^T, \quad (4.12a)$$

$$\bar{Z}_2(z_2) = (1 \quad z_2^{-1} \quad z_2^{-2} \quad \cdots \quad z_2^{-d_{11}+1})^T. \quad (4.12b)$$

The polyphase components of the entire analysis bank, can now be written as a polynomial matrix of size  $N \times N$ :

$$\mathbf{H}_p(z_1, z_2) = \begin{pmatrix} \text{vec} [\mathbf{H}_{p0}(z_1, z_2)]^T \\ \text{vec} [\mathbf{H}_{p1}(z_1, z_2)]^T \\ \vdots \\ \text{vec} [\mathbf{H}_{pN-1}(z_1, z_2)]^T \end{pmatrix}. \quad (4.13)$$

The  $\mathbf{H}_{pi}$ 's are given by Eq. (4.11), and the  $\text{vec} [\cdot]$  operator creates a column vector out of a matrix by stacking the columns on top of one another (*i.e.*, for a matrix with  $N$  rows, its element  $a_{ij}$  becomes element  $a_{i+jN}$  of the column vector) [GRA81]. By using the fact that  $\text{vec} [\mathbf{X}\mathbf{Y}\mathbf{Z}]^T = \text{vec} [\mathbf{Y}]^T \cdot \mathbf{Z} \otimes \mathbf{X}^T$  [GRA81] ( $\otimes$  denotes the Kronecker matrix-product), Eq. (4.10) can be written as

$$\begin{aligned} H_i(z_1, z_2) &= \text{vec} [H_i(z_1, z_2)]^T \\ &= \text{vec} \left[ \bar{Z}_1(z_1)^T \mathbf{H}_{pi} \begin{pmatrix} z_1^{d_{00}} & z_1^{d_{01}} z_2^{d_{11}} \end{pmatrix} \bar{Z}_2(z_2) \right]^T \\ &= \text{vec} \left[ \mathbf{H}_{pi} \begin{pmatrix} z_1^{d_{00}} & z_1^{d_{01}} z_2^{d_{11}} \end{pmatrix} \right]^T \bar{Z}_2(z_2) \otimes \bar{Z}_1(z_1). \end{aligned} \quad (4.14)$$

Thus, by using Eqs. (4.13) and (4.14), the analysis filter bank can be expressed in terms of its polyphase components as

$$\bar{H}(z_1, z_2) = \begin{pmatrix} H_0(z_1, z_2) \\ H_1(z_1, z_2) \\ \vdots \\ H_{N-1}(z_1, z_2) \end{pmatrix} = \mathbf{H}_p \begin{pmatrix} z_1^{d_{00}} & z_1^{d_{01}} z_2^{d_{11}} \end{pmatrix} \bar{Z}_2(z_2) \otimes \bar{Z}_1(z_1). \quad (4.15)$$

Note that

$$\bar{Z}_2(z_2) \otimes \bar{Z}_1(z_1) = (1 \quad z_1^{-1} \quad \dots \quad z_1^{-d_{00}+1} \quad z_2^{-1} \quad \dots \quad z_1^{-d_{00}+1} z_2^{-d_{11}+1})^T.$$

The representation of Eq. (4.15) will be used next when considering the input/output relationship of the entire subband coding scheme; the analysis, subsampling, upsampling and the synthesis.

*Example 3* For the hexagonal subsampling the filter bank is described through its polyphase components as

$$\bar{H}(z_1, z_2) = \begin{pmatrix} H_{p0,0,0}(z_1^2, z_1 z_2^2) & \dots & H_{p0,1,1}(z_1^2, z_1 z_2^2) \\ \vdots & \dots & \vdots \\ H_{p3,0,0}(z_1^2, z_1 z_2^2) & \dots & H_{p3,1,1}(z_1^2, z_1 z_2^2) \end{pmatrix} \begin{pmatrix} 1 \\ z_1^{-1} \\ z_2^{-1} \\ (z_1 z_2)^{-1} \end{pmatrix}.$$

### 4.3 Subband analysis and synthesis

An input signal,  $X(z_1, z_2)$ , is passed through the analysis filters and subsequently sub- and upsampled. In analogy with Eq. (4.8), this gives

$$\begin{aligned} \bar{Y}(z_1, z_2) &= \frac{1}{N} \sum_{m=0}^{d_{00}-1} \sum_{n=0}^{d_{11}-1} \bar{H} \left( W_N^{-d_{11}m} z_1, W_N^{-(d_{00}n-d_{01}m)} z_2 \right) \\ &\quad \cdot X \left( W_N^{-d_{11}m} z_1, W_N^{-(d_{00}n-d_{01}m)} z_2 \right) \\ &= \frac{1}{N} \mathbf{H}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) \sum_{m=0}^{d_{00}-1} \sum_{n=0}^{d_{11}-1} \bar{Z}_2 \left( W_N^{-(d_{00}n-d_{01}m)} z_2 \right) \otimes \bar{Z}_1 \left( W_N^{-d_{11}m} z_1 \right) \\ &\quad \cdot X \left( W_N^{-d_{11}m} z_1, W_N^{-(d_{00}n-d_{01}m)} z_2 \right). \end{aligned} \tag{4.16}$$

Note that the polyphase matrix,  $\mathbf{H}_p$ , is unaffected by the modulation since its elements are polynomials in powers of  $(z_1^{d_{00}})$  and  $(z_1^{d_{01}} z_2^{d_{11}})$ . It is shown for a monomial with powers  $u_1$  and  $u_2$ , by using Eq. (4.3):

$$\begin{aligned} &\left( W_N^{-d_{11}m} z_1 \right)^{-d_{00}u_1} \left( W_N^{-d_{11}m} z_1 \right)^{-d_{01}u_2} \left( W_N^{-d_{00}n} z_2 \right)^{-d_{11}u_2} = \\ &W_N^{(Nu_1m+Nu_2n)} z_1^{-(d_{00}u_1+d_{01}u_2)} z_2^{-d_{11}u_2} = \left( z_1^{d_{00}} \right)^{-u_1} \left( z_1^{d_{01}} z_2^{d_{11}} \right)^{-u_2}. \end{aligned} \tag{4.17}$$

Analogously to Eq. (4.15), the bank of synthesis filters,  $\bar{G}(z_1, z_2)$ , can be defined as a column vector of  $N$  impulse responses:

$$\bar{G}(z_1, z_2) = z_1^{-d_{00}+1} z_2^{-d_{11}+1} \mathbf{G}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) \bar{Z}_2(z_2^{-1}) \otimes \bar{Z}_1(z_1^{-1}), \quad (4.18)$$

where the column vectors  $\bar{Z}_1$  and  $\bar{Z}_2$  are given by Eqs. (4.12a) and (4.12b). Note that

$$z_1^{-d_{00}+1} z_2^{-d_{11}+1} \bar{Z}_2(z_2^{-1}) \otimes \bar{Z}_1(z_1^{-1}) = \bar{Z}_2(z_2) \otimes \bar{Z}_1(z_1) \mathbf{J},$$

where  $\mathbf{J}$  is a matrix with unit elements along the antidiagonal (*confer* Eq. (4.15)).

Given Eqs. (4.16) and (4.18), the output from the system,  $\hat{X}(z_1, z_2)$ , can be expressed as

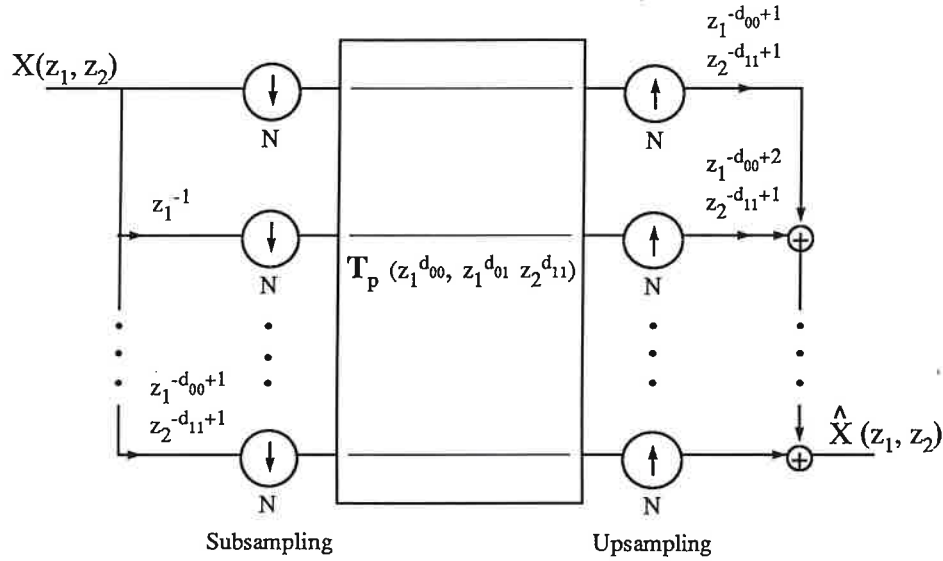
$$\begin{aligned} \hat{X}(z_1, z_2) &= \bar{G}(z_1, z_2)^T \bar{Y}(z_1, z_2) = \frac{1}{N} z_1^{-d_{00}+1} z_2^{-d_{11}+1} \bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T \\ &\quad \mathbf{G}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right)^T \mathbf{H}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) \\ &\quad \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} \bar{Z}_2 \left( W_N^{-(d_{00}l-d_{01}k)} z_2 \right) \otimes \bar{Z}_1 \left( W_N^{-d_{11}k} z_1 \right) \\ &\quad \cdot X \left( W_N^{-d_{11}k} z_1, W_N^{-(d_{00}l-d_{01}k)} z_2 \right). \end{aligned} \quad (4.19)$$

The entire system as described by Eq. (4.19) can be illustrated by a block diagram as in Fig. 4.3, where the transfer matrix  $\mathbf{T}_p$  is defined as

$$\mathbf{T}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) = \mathbf{G}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right)^T \mathbf{H}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right). \quad (4.20)$$

In order to eliminate the aliasing components of  $X(z_1, z_2)$  in Eq. (4.19), so that the system becomes shift invariant, the following condition is necessary and sufficient for the transfer matrix:

$$\bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T \mathbf{T}_p \left( z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}} \right) = T(z_1, z_2) \bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T, \quad (4.21)$$



*Figure 4.3 The system described by Eq. (4.19) with  $\mathbf{T}_p$  defined in Eq. (4.20). If the box would be replaced by short-circuits of each branch, it would correspond to  $\mathbf{T}_p = \mathbf{I}$ . Note that the product of delays is the same along all horizontal branches.*

where  $T(z_1, z_2)$  is a scalar polynomial, or, in the case of infinite impulse-response systems, a ratio of two scalar polynomials (i.e.,  $T(z_1, z_2) = R(z_1, z_2)/Q(z_1, z_2)$ ). The condition means that  $\bar{Z}_2(z_2^{-1}) \otimes \bar{Z}_1(z_1^{-1})$  is an eigenvector of  $\mathbf{T}_p(z_1^{d_{00}}, z_1^{d_{01}}, z_2^{d_{11}})^T$  with the eigenvalue  $T(z_1, z_2)$ .

First the sufficiency of the condition is shown, i.e., that all aliasing components are indeed cancelled. If  $\bar{Z}_1(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T$  is moved inside the summations in Eq. (4.19), then for fixed  $m$  and  $n$  we have

$$\begin{aligned}
 & \left( \bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T \right) \left( \bar{Z}_2 \left( W_N^{-(d_{00}n - d_{01}m)} z_2 \right) \otimes \bar{Z}_1 \left( W_N^{-d_{11}m} z_1 \right) \right) \\
 &= \underbrace{\sum_{i=0}^{d_{11}-1} W_N^{-(d_{00}n - d_{01}m)i}}_{=0, n \neq 0} \underbrace{\sum_{j=0}^{d_{00}-1} W_N^{-d_{11}mj}}_{=0, m \neq 0} = \begin{cases} N, & m = n = 0 \\ 1, & 1 \leq m < d_{00} \\ 0, & \\ 1, & 1 \leq n < d_{11} \end{cases} \quad (4.22)
 \end{aligned}$$

The above used the fact that the product  $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ , if the matrix products are defined [GRA81].

The necessity of Eq. (4.21) is shown as follows. Let the right-hand side of the equation be replaced by a general row vector,  $\bar{V}(z_1, z_2)^T$ . If this vector is moved inside the summation of Eq. (4.19), then for fixed  $m$  and  $n$  the condition for alias cancellation yields that

$$\sum_{i=0}^{d_{00}-1} \sum_{j=0}^{d_{11}-1} V_{i+jd_{00}}(z_1, z_2) \left(W_N^{-d_{11}m} z_1\right)^{-i} \left(W_N^{-(d_{00}n-d_{01}m)} z_2\right)^{-j} = \begin{cases} \mathcal{V}(z_1, z_2), & m = n = 0 \\ 1 \leq m < d_{00} \\ 0, \\ 1 \leq n < d_{11} \end{cases} \quad (4.23)$$

where  $V_{i+jd_{00}}(z_1, z_2)$  are the polynomial elements of  $\bar{V}$ , and  $\mathcal{V}(z_1, z_2)$  is an arbitrary polynomial. In order to get cancellations, the elements of  $\bar{V}$  has to be of the form  $V_{i+jd_{00}}(z_1, z_2) = \Upsilon_{i+jd_{00}}(z_1, z_2) z_1^i z_2^j$ , for which Eq. (4.23) becomes

$$\sum_{i=0}^{d_{00}-1} \sum_{j=0}^{d_{11}-1} \Upsilon_{i+jd_{00}}(z_1, z_2) W_N^{(d_{11}mi+d_{00}nj-d_{01}mj)} = \begin{cases} \mathcal{V}(z_1, z_2), & m = n = 0 \\ 1 \leq m < d_{00} \\ 0, \\ 1 \leq n < d_{11} \end{cases} \quad (4.24)$$

This condition can be met only if all the  $\Upsilon$ 's are equal, in which case  $\Upsilon(z_1, z_2) = \mathcal{V}(z_1, z_2)$  and the cancellation of terms is given by Eq. (4.22). This, in turn, means that  $\bar{V}(z_1, z_2)^T$  can be written as the right-hand side of Eq. (4.21) where  $T(z_1, z_2) = \mathcal{V}(z_1, z_2)$ .

Note that the condition given in Eq. (4.21) does not stipulate a certain matrix structure. In the case of separable subsampling (*i.e.*,  $d_{01} = 0$ ), the structure has been shown to be block pseudo-circulant in  $z_2$  where each block is pseudo-circulant in  $z_1$  [LIU88]. This form is generally not valid for non-separable subsampling, except in its most restricted form:  $T(z_1, z_2) \mathbf{I}_{N \times N}$ .

Example 4 For the hexagonal subsampling, the following form of  $\mathbf{T}_p$  is sufficient to meet the condition in Eq. (4.21):

$$\mathbf{T}_p(z_1, z_2) = \begin{pmatrix} T_0(z_1, z_2) & T_1(z_1, z_2) & T_2(z_1, z_2) & T_3(z_1, z_2) \\ T_1(z_1, z_2)z_1^{-1} & T_0(z_1, z_2) & T_3(z_1, z_2)z_1^{-1} & T_2(z_1, z_2) \\ T_3(z_1, z_2)z_2^{-1} & T_2(z_1, z_2)z_1z_2^{-1} & T_0(z_1, z_2) & T_1(z_1, z_2) \\ T_2(z_1, z_2)z_2^{-1} & T_3(z_1, z_2)z_2^{-1} & T_1(z_1, z_2)z_1^{-1} & T_0(z_1, z_2) \end{pmatrix}$$

which gives

$$\begin{pmatrix} 1 & z_1 & z_2 & z_1z_2 \end{pmatrix} \mathbf{T}_p \begin{pmatrix} z_1^2 & z_1z_2^2 \end{pmatrix} = T(z_1, z_2) \begin{pmatrix} 1 & z_1 & z_2 & z_1z_2 \end{pmatrix}$$

where

$$\begin{aligned} T(z_1, z_2) &= T_0(z_1^2, z_1z_2^2) + z_1^{-1}T_1(z_1^2, z_1z_2^2) \\ &\quad + z_2^{-1}T_2(z_1^2, z_1z_2^2) + z_1^{-1}z_2^{-1}T_3(z_1^2, z_1z_2^2). \end{aligned}$$

After cancellation of the aliasing terms in Eq. (4.19), the system is linear and shift-invariant which can be represented by a (scalar) transfer function. The output from the entire system is thus a filtered version of the input signal

$$\hat{X}(z_1, z_2) = z_1^{-d_{00}+1} z_2^{-d_{11}+1} T(z_1, z_2) X(z_1, z_2), \quad (4.25).$$

In order to get perfect reconstruction, where the output from the system is a perfect replica of the input signal, it is necessary and sufficient that

$$T(z_1, z_2) = z_1^{-k} z_2^{-l}, \quad (4.26)$$

i.e.,  $T(z_1, z_2)$  is a monic monomial. Hence, the transfer function only causes a shift of the input ( $k + d_{00} - 1$  steps horizontally and  $l + d_{11} - 1$  steps vertically), and it does not produce any spectral, phase nor amplitude distortion! A similar result, given in the Fourier domain, appears in [VIS88]. In the remainder of this chapter, we will concentrate on perfect reconstruction systems obtained by transfer matrices of the form  $\mathbf{T}_p(z_1, z_2) = T(z_1, z_2) \mathbf{I}_{N \times N}$ .

#### 4.4 Two-dimensional perfect reconstruction FIR filter banks

In the following sections, we are going to look at design structures which meet the condition for perfect reconstruction, as derived in the previous section. A desirable property for the analysis and the synthesis banks is that they both should consist of finite impulse-response filters. This property is met *if and only if* the determinant of the polyphase matrix of a filter bank is a monomial. In this case, we do not have to be concerned with stability issues which are most intricate in the two-dimensional case [BOS82]. An appealing way of constructing FIR filter banks is through a cascade of simple building blocks. That way the filter dimensions of a system may be iteratively enlarged by multiplying the system with one or more blocks. However, this procedure requires that all the systems of interest can be constructed as products of simple factors (building blocks). In the one-dimensional case this is straightforward since all polynomials, which appear as elements in a polyphase matrix, can be decomposed into first order factors over the complex field. The same is generally not true for two-dimensional polynomials. However, when both the analysis and the synthesis filter banks are FIR the polyphase matrices can in fact be factorized! To show this, we restate a result from [MOR77, Theorem 4.2]: If a polynomial matrix,  $\mathbf{R}(z_1, z_2)$ , has a determinant that can be factorized,

$$\det(\mathbf{R}(z_1, z_2)) = \prod_{i=1}^k \mathcal{R}_i(z_1, z_2), \quad (4.27a)$$

where  $\mathcal{R}_i(z_1, z_2)$  are arbitrary polynomials, then the matrix can be factorized so that

$$\mathbf{R}(z_1, z_2) = \prod_{i=1}^k \mathbf{R}_i(z_1, z_2), \quad (4.27b)$$

and

$$\det(\mathbf{R}_i(z_1, z_2)) = \mathcal{R}_i(z_1, z_2). \quad (4.27c)$$

The coefficients of the polynomial matrix-elements can belong to any field (of our particular interest, the field of real numbers), which was shown by Guiver and Bose in [GUI82]. The factorization is however not unique, and Youla and Gnavi has shown that it does not extend to systems with three or more dimensions [YOU79]. As mentioned, FIR systems have  $\det (\mathbf{H}_p(z_1, z_2)) = c z_1^{-m_1} z_2^{-m_2}$ . We therefore know that the above factorization exists and that all factors have the determinants  $\mathcal{R}_i$  equal to either  $c z_1^{-1}$  or  $c z_2^{-1}$ , for arbitrary constants  $c$ .

System type	Free variables	$N = 4, m_1 = m_2 = 1$
Unconstrained system	$N^2 (m_1 + 1)(m_2 + 1)$	64
Analysis and synthesis FIR	$N^2 + N(N - 1)(m_1 + m_2)$	40
Invertible FIR system ( $\det \mathbf{D} \neq 0$ )	$N^2 + (2N - 1)(m_1 + m_2) - 1$	29
Paraunitary FIR system	$(N-1)(N + 2(m_1 + m_2)) / 2$	12

*Table 4.1 Free variables for polyphase matrices of various types. The unconstrained type gives the maximally available number. Note that  $\mathbf{D}$  refer to the inverted system in Eq. (4.41).*

We shall obtain perfect reconstruction by two means: paraunitary systems and systems with invertible state-space description. (By the latter we mean that the polyphase matrix is taken to be the transfer function, as given by some state-space description, and the synthesis system results from the inverse transfer function.) Design of filter banks based on both types of systems will be studied in the next two sections. Before that we shall determine how constrained these systems types are. Generally speaking, the more free variables available for the design, the closer, or more easily, can design goals be met. In Table 4.1 we show the number of free variables for a polyphase matrix of size  $N \times N$  whose elements have highest power  $m_1$  in  $z_1$  and  $m_2$  in  $z_2$ . The system with FIR filters in both banks is constrained only through its determinant which should be a monomial. The determinant of the polyphase matrix is a polynomial with  $N(Nm_1 + 1)(Nm_2 + 1)$  terms out of which

all but one term are constrained to zero. The expressions in the table for invertible and paraunitary systems are derived in the sections that follow.

#### 4.5 Paraunitary systems for perfect reconstruction filter banks

If we wish to have impulse responses which are equal in both filter banks (within a reversal in the directions  $n_1$  and  $n_2$ ), then we should impose a paraunitary condition on the polyphase matrix. This given by [VAI87b] for the two-dimensional case

$$\mathbf{H}_p(z_1^{-1}, z_2^{-1})^\dagger \mathbf{H}_p(z_1, z_2) = \mathbf{I}. \quad (4.28)$$

( $\dagger$  means transposition followed by complex conjugation of the coefficients, but not conjugation of  $z_1$  and  $z_2$ .) Consequently, the synthesis filters can be chosen as  $\mathbf{G}_p(z_1, z_2) = F(z_1, z_2)\mathbf{H}_p(z_1^{-1}, z_2^{-1})$ , where  $F(z_1, z_2)$  is a monic monomial of powers so that  $\mathbf{G}_p$  is causal. We will limit ourselves to polynomial matrices with real-valued coefficients (so,  $\dagger \rightarrow T$ , regular matrix transposition). Note that Eq. (4.28) holds for  $\mathbf{H}_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})$  as well, as needed for Eq. (4.21). (A paraunitary matrix is, by definition, unitary on the unit bi-circles,  $z_1 = e^{-j\omega_1}$  and  $z_2 = e^{-j\omega_2}$ . Thus, it will not be affected by integer powers of  $z_1$  and  $z_2$ .)

##### 4.5.1 Filter bank design by a cascade-structure

For the purpose of constructing paraunitary systems of any degree, we will seek a design structure of the matrix  $\mathbf{H}_p(z_1, z_2)$ . The structure is a straightforward generalization of the one-dimensional one given in [VAI89].

The one-dimensional design structure is

$$\mathbf{H}_p(z) = \mathbf{H}_0 \prod_{i=0}^{w-1} \{\mathbf{I} - (1 - z^{-1})\bar{u}_i \bar{u}_i^T\}, \quad (4.29)$$

where  $\mathbf{H}_0$  is orthogonal and  $\bar{u}_i$  is a real-valued column vector with unit norm. This structure yields all one-dimensional paraunitary systems. The corresponding two-dimensional structure, which is not complete, is then given by

$$\mathbf{H}_p(z_1, z_2) = \mathbf{H}_0 \prod_{i=0}^{w-1} \{\mathbf{I} - (1 - z_1^{-1})\bar{u}_i\bar{u}_i^T\} \{\mathbf{I} - (1 - z_2^{-1})\bar{v}_i\bar{v}_i^T\} \quad (4.30)$$

where  $\mathbf{H}_0$  is an orthogonal matrix of size  $N \times N$ ,  $\bar{u}_i^T \bar{u}_i = \{1, \text{ or } 0\}$  and  $\bar{v}_i^T \bar{v}_i = \{1, \text{ or } 0\}$ ,  $\forall i \in [0, w)$ , and all column-vectors  $\bar{u}_i$  and  $\bar{v}_i$  have length  $N$ . Note that we have to allow  $\bar{u}_i^T \bar{u}_i = 0$  and  $\bar{v}_i^T \bar{v}_i = 0$  so that we can get neighboring factors in the same variable. The system may have different orders in  $z_1$  and  $z_2$ , say  $w_1$  and  $w_2$ , respectively, which is accommodated by having  $w - w_1$  factors in Eq. (4.30) where  $\bar{u}_i = \bar{0}$  and  $w - w_2$  factors where  $\bar{v}_i = \bar{0}$ . Note that the entire system is non-separable since the factors do not commute.

The orthogonal matrix,  $\mathbf{H}_0$ , in Eq. (4.30) may in fact be arbitrarily applied by pre- or post-multiplication, so that all the systems yielded by post-multiplication may equally well be found by pre-multiplication. Consider, the minimal two-dimensional cascade

$$\mathbf{H}_p(z_1, z_2) = \{\mathbf{I} - (1 - z_1^{-1})\bar{u}_1\bar{u}_1^T\} \{\mathbf{I} - (1 - z_2^{-1})\bar{v}_1\bar{v}_1^T\} \mathbf{H}_0. \quad (4.31)$$

For  $\bar{u}_1 = \mathbf{H}_0 \bar{u}_0$  and  $\bar{v}_1 = \mathbf{H}_0 \bar{v}_0$ , we get

$$\begin{aligned} \mathbf{H}_p(z_1, z_2) &= \{\mathbf{I} - (1 - z_1^{-1})\mathbf{H}_0 \bar{u}_0\bar{u}_0^T \mathbf{H}_0^T\} \{\mathbf{H}_0 - (1 - z_1^{-1})\mathbf{H}_0 \bar{v}_0\bar{v}_0^T \mathbf{H}_0^T \mathbf{H}_0\} \\ &= \{\mathbf{H}_0 - (1 - z_1^{-1})\mathbf{H}_0 \bar{u}_0\bar{u}_0^T \mathbf{H}_0^T \mathbf{H}_0\} \{\mathbf{I} - (1 - z_1^{-1})\bar{v}_0\bar{v}_0^T\} \\ &= \mathbf{H}_0 \{\mathbf{I} - (1 - z_1^{-1})\bar{u}_0\bar{u}_0^T\} \{\mathbf{I} - (1 - z_2^{-1})\bar{v}_0\bar{v}_0^T\}. \end{aligned} \quad (4.32)$$

By continuing this procedure,  $\mathbf{H}_0$  may be moved from right to left through the whole cascade in Eq. (4.30), which completes the proof.

The cascade structure resembles the factorization that we would expect from Eq. (4.27b). However, we do not know at this point just how complete this design structure is. Completeness aside, the cascade structure has also other shortcomings. A certain length of the cascade, say  $w_1$  factors in  $z_1$  and  $w_2$  in  $z_2$ , will give systems of highest orders anywhere in the range of  $(1 \dots w_1)$  in  $z_1$  and  $(1 \dots w_2)$  in  $z_2$ . Every pair of orthogonal vectors  $\bar{u}_i$  will lower the degree in  $z_1$  and, analogously, orthogonal vectors  $\bar{v}_i$  lower the degree in  $z_2$ . Thus, to design a paraunitary matrix of a given order, one has to allow cascades of much higher order which can be reduced to the desired order by appropriate orthogonality constraints. If the desired order is, say,  $m_1$  in  $z_1$  and  $m_2$  in  $z_2$  then the cascade has to be allowed to include up to  $w_1 = m_1 + N - 1$  factors in  $z_1$  and  $w_2 = m_2 + N - 1$  in  $z_2$ , where  $N$  is the length of the column vectors.

The advantages with the structure is that the filter bank design can be performed iteratively, with one factor at a time. Although the structure may not complete over two-dimensional paraunitary systems even for complex valued factors, we believe that it may nevertheless yield useful non-separable filters.

#### 4.5.2 Design based on paraunitary state-space description

Since the previous structure may not generate all possible FIR paraunitary systems of a given degree, we will seek to design polyphase matrices from the state-space description which may produce filter banks not obtainable with the previous method. The derivation is a generalization of [DOG88] where it is performed for one-dimensional systems.

The transfer function of a two-dimensional system can be written as [ROE75,

KUN77, BOS82]:

$$\mathbf{H}_p(z_1, z_2) = \mathbf{D} + \mathbf{C} \left\{ \begin{pmatrix} z_1 \mathbf{I}_{m_1} & \mathbf{0} \\ \mathbf{0} & z_2 \mathbf{I}_{m_2} \end{pmatrix} - \mathbf{A} \right\}^{-1} \mathbf{B}, \quad (4.33)$$

where  $\mathbf{A}$  is a  $m \times m$  matrix,  $\mathbf{D}$  has size  $N \times N$ ,  $\mathbf{C}$  has size  $N \times m$ , and  $\mathbf{B}$  has size  $m \times N$ , and  $m = m_1 + m_2$ .

For the system to consist of FIR filters, it is necessary that  $\mathbf{A}$  is lower [upper] triangular with all elements on the main diagonal equal to zero. That is,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 0 \end{pmatrix}. \quad (4.34)$$

(‘\*’ marks elements which are not required to be zero.) We form a  $p \times p$  matrix,  $\mathbf{R}$ , which consists of  $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$  as sub-matrices:

$$\mathbf{R} = \begin{matrix} & \begin{matrix} N & m \end{matrix} \\ \begin{matrix} m \\ N \end{matrix} & \begin{pmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{D} & \mathbf{C} \end{pmatrix} \end{matrix}. \quad (4.35)$$

When  $\mathbf{R}$  is orthogonal (or unitary, in the complex-valued case) then the system in Eq. (4.33) is paraunitary.

Bose gives this result in a slightly different form [BOS82]. The system of Eq. (4.35) is transformed so that the matrices  $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$  become polynomial matrices in one variable,  $[\mathbf{A}(z_2), \mathbf{B}(z_2), \mathbf{C}(z_2), \mathbf{D}(z_2)]$ . The transformed two-dimensional system is then paraunitary if  $\mathbf{R}(z_2)$  is paraunitary.

Doğanata *et alii* give a design structure for orthogonal matrices,  $\mathbf{R}$ , which have the sub-matrix  $\mathbf{A}$  of the form given by Eq. (4.34) [DOG88]:

$$\mathbf{R} = \prod_{i=p-2}^0 \prod_{j=p-1}^{i+1} \Theta_{i,j} \quad (4.36a)$$

and

$$\Theta_{i,j} = \mathbf{I} \quad \text{for} \quad 0 \leq i \leq m-1, \quad N+i \leq j \leq p-1. \quad (4.36b)$$

In the above,  $p = N + m$  and  $\Theta_{i,j}$  is a rotation matrix of the form

$$\Theta_{i,j} = \begin{matrix} & \begin{matrix} 0 & 1 & & i & & j & & p-1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ p-1 \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cos \theta_{i,j} & \cdots & -\sin \theta_{i,j} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sin \theta_{i,j} & \cdots & \cos \theta_{i,j} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{matrix}. \quad (4.37)$$

This gives a design structure which appears to cover all possible paraunitary systems with highest degree  $m_1$  in  $z_1$  and  $m_2$  in  $z_2$ . For this structure, Doğanata *et alii* [DOG88] give the number of free variable (angles  $\theta$  in this case)

$$F_p = \binom{m+N}{2} - \binom{m+1}{2} = \frac{1}{2}(N+2m)(N-1) = \frac{1}{2}(N+2(m_1+m_2))(N-1). \quad (4.38)$$

Note that the desired form for  $\mathbf{A}$ , as given in Eq. (4.34), cannot in general be obtained by a similarity transform. (A similarity transform means that for some orthogonal matrix  $\mathbf{S}$ , the realization  $[\mathbf{S}^T \mathbf{A}' \mathbf{S}, \mathbf{S}^T \mathbf{B}', \mathbf{C}' \mathbf{S}, \mathbf{D}]$  gives the same transfer function as  $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$ , *i.e.*, the two realizations are similar [KAI80].) The reasons are, firstly, that the sub-matrix,  $\mathbf{A}$ , may be non-singular for a given orthogonal matrices  $\mathbf{R}$  in Eq. (4.35). Secondly, for two-dimensional state-space description as in Eq. (4.33),  $\mathbf{S}$  is restricted to be of the form [ROE75]

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{m_1}^0 & 0 \\ 0 & \mathbf{S}_{m_2}^1 \end{pmatrix}. \quad (4.39)$$

The subscripts denote the dimensions of the square sub-matrices  $\mathbf{S}^0$  and  $\mathbf{S}^1$ , which both are orthogonal. So, this restricted form of  $\mathbf{S}$  may not be sufficient to triangularize most singular matrices. Hence, it is preferable to design orthogonal matrices which already have zeroes in the upper-right corner.

Next we will look at a limited case which forms the two-dimensional generalization of so called *lapped orthogonal transforms*, or simply LOT's. A LOT is a paraunitary filter bank with one-dimensional filters whose lengths are twice the subsampling factor, *i.e.*,  $2N$  [CAS85, MAL86, MAL88]. For a LOT, all polyphase components are therefore first order (linear) polynomials. (Such a polynomial matrix is referred to as a matrix pencil in systems theory.) The analogous two-dimensional system should have polyphase components with highest order  $z_1^{-1}z_2^{-1}$ . That means that the finite impulse response has support by an area consisting of four unit cells on the input lattice in a  $2 \times 2$  arrangement, as illustrated in Fig. 4.4. The corresponding design structure is given by

$$\mathbf{H}_p(z_1, z_2) = \mathbf{D} + \begin{pmatrix} \bar{c}_0 & \bar{c}_1 \end{pmatrix} \begin{pmatrix} z_1^{-1} & 0 \\ az_1^{-1}z_2^{-1} & z_2^{-1} \end{pmatrix} \begin{pmatrix} \bar{b}_0^T \\ \bar{b}_1^T \end{pmatrix}, \quad (4.40)$$

where  $a$  is the only non-zero element of  $\mathbf{A}$ , and the  $b$ 's and  $c$ 's are column vectors of length  $N$ . The condition in Eq. (4.36b) then is simplified to  $\Theta_{0,N} = \Theta_{0,N+1} = \Theta_{1,N+1} = \mathbf{I}$ , or equivalently,  $\theta_{0,N} = \theta_{0,N+1} = \theta_{1,N+1} = 0$  as given by Eq. (4.37).

**Example 5** *In order to illustrate the design method of Eq. (4.40) for the case of two-dimensional LOT's, the hexagonal subsampling case ( $N = 4$ ) is chosen. Thus,  $p = 6$  and  $\mathbf{R}$  is given by*

$$\mathbf{R} = \Theta_{4,5} \Theta_{3,5} \Theta_{3,4} \Theta_{2,5} \Theta_{2,4} \Theta_{2,3} \Theta_{1,4} \Theta_{1,3} \Theta_{1,2} \Theta_{0,3} \Theta_{0,2} \Theta_{0,1}$$

*To make the example clearer, the rotation matrices  $\Theta_{i,j}$  can be replaced by scaled ones (divided by  $\cos \theta_{i,j}$ ) so that  $\cos \theta_{i,j} = 1$  and  $\sin \theta_{i,j} = a_{i,j}$  (assuming that*

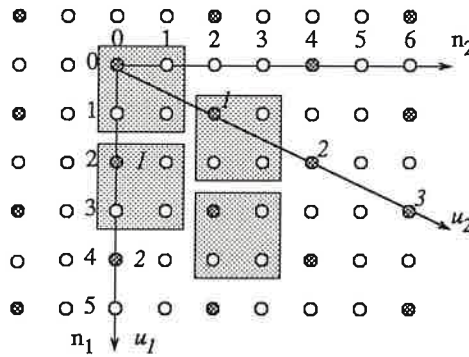
$\theta_{i,j} \neq (2k+1)\pi/2$ ). With this approximation of  $\mathbf{R}$  we get

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ -a_{1,4} & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & -a_{0,1} \\ a_{0,1} - a_{0,2}a_{1,2} - a_{0,3}a_{1,3} & 1 + a_{0,1}(a_{0,2}a_{1,2} + a_{0,3}a_{1,3}) \\ -a_{0,2} & -a_{0,3} \\ a_{0,2}a_{0,3}a_{1,3} - a_{1,2} & -a_{1,3} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -a_{2,4} & -a_{2,5} \\ a_{2,4}a_{2,5}a_{3,5} - a_{3,4} & a_{3,5} \\ 1 + (a_{2,4}a_{2,5} + a_{3,4}a_{3,5})a_{4,5} & -a_{4,5} \\ a_{4,5} - a_{2,4}a_{2,5} - a_{3,4}a_{3,5} & 1 \end{pmatrix}.$$

The matrix  $\mathbf{D}$  has been left out for reasons of conciseness. The area of support for the impulse responses is shown in Fig. 4.4.



*Figure 4.4* Region of support for the impulse responses of two-dimensional LOT's with hexagonal subsampling.

The outlined design method is less tractable for filter bank design than the cascade structure. The issue is that, in this method, one first designs an orthogonal matrix. This matrix has then to be partitioned into the sub-matrices,  $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$ , which can be subsequently used in the state-space description to find the polyphase matrix. In short, the effect on the filter properties of a particular angle in  $\mathbf{R}$  may be hard to foresee.

#### 4.6 Non-paraunitary systems for perfect reconstruction filter banks

Next we will search for analysis and synthesis FIR structures which are not paraunitary. Paraunitary systems are appealing mainly because the analysis and synthesis filter banks are the same. If the analysis filters perform well, so do the synthesis filters. However, this property is obtained at a cost: the system is highly constrained. Since none of these constraints pertain to the frequency responses of the filters, we may end up with poor performing filters only because we impose constraints which simplify the design procedure. By going over to non-paraunitary systems, less structural conditions are imposed. In return, one has to accept the occurrence of undesirable effects in general perfect reconstruction systems, where the synthesis system is the inverse of the analysis system. The first potential problem is that the synthesis system may not correspond to a filter bank with clearly defined frequency characteristics. Secondly, the synthesis filters, it is feared, may be of vastly larger size than the analysis filters. The first problem can be avoided by specifying design goals for both filter banks. However, the design will be more complex than in the paraunitary case since, in effect, two filter banks are to be designed simultaneously. Then we shall see that for the proposed design structure, which is mildly constrained, the size of the largest synthesis filter will not exceed that of the largest analysis filter. We also include a discussion on filter bank design based on the Smith form of a polynomial matrix.

##### 4.6.1 Design based on state-space description of inverse system

Given the state-space description of an analysis filter bank, as in Eq. (4.33), the synthesis filter bank for perfect reconstruction could be taken simply as the transpose of the inverse system, *i.e.*,  $\mathbf{G}_p(z_1, z_2) = \mathbf{H}_p(z_1, z_2)^{-T}$ . The inverse of the

“forward” system in Eq. (4.33) is given by

$$\mathbf{H}_p(z_1, z_2)^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{C} \left\{ \begin{pmatrix} z_1 \mathbf{I}_{m_1} & \mathbf{0} \\ \mathbf{0} & z_2 \mathbf{I}_{m_2} \end{pmatrix} - \mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C} \right\}^{-1} \mathbf{B}\mathbf{D}^{-1}, \quad (4.41)$$

provided that  $\mathbf{D}$  is non-singular [KAI80]. If the “forward” system is FIR, the inverse system will be FIR as well if

$$\mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 0 \end{pmatrix} = \mathbf{L}_\Delta. \quad (4.42)$$

Let  $\mathbf{C} = \mathbf{D} \mathbf{E}$ , where  $\mathbf{E}$  has dimensions  $N \times m$ , just as  $\mathbf{C}$ . Hence, Eq. (4.42) becomes  $\mathbf{B}\mathbf{E} = \mathbf{L}_\Delta$  which can be written more explicitly as

$$(\bar{b}_0 \ \bar{b}_1 \ \cdots \ \bar{b}_{m-1})^T (\bar{e}_0 \ \bar{e}_1 \ \cdots \ \bar{e}_{m-1}) = \mathbf{L}_\Delta, \quad (4.43a)$$

$$\Rightarrow \quad \bar{b}_i^T \bar{e}_j = 0, \quad \text{for } j \geq i \text{ and } \forall i \in (0, m]. \quad (4.43b)$$

In the above, the  $\bar{b}$ 's and  $\bar{e}$ 's are column vectors of length  $N$ . At this point, we can note that the highest powers of the inverse system in Eq. (4.41) are no higher than those of the “forward” system, namely  $z_1^{-m_1}$  and  $z_2^{-m_2}$ . These powers are given by the determinant of the diagonal “delay” matrix. (All cofactors of the matrix have positive powers of  $z_1$  and  $z_2$ , which are of strictly lower powers than the determinant. The highest powers of the inverse thus come from the constant terms of the cofactors multiplied by the inverse of the determinant.) In conclusion, the synthesis filters will be no larger than the analysis filters.

A marginal observation: if  $\mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$  then the inverse system is given by

$$\mathbf{H}_p(z_1, z_2)^{-1} = 2\mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{H}_p(z_1, z_2)\mathbf{D}^{-1}. \quad (4.44)$$

The inverse system is thus obtained through a transformation of the “forward” system. In particular, when  $\mathbf{D}$  is orthogonal this limited system gives the inverse without any matrix-inversion.

Let us now look at a system which corresponds to the 2-D LOT in the paraunitary case. Since the derivation has been done for the general system we only state the result.

$$\mathbf{H}_p(z_1, z_2) = \mathbf{D} + \mathbf{D} \begin{pmatrix} \bar{\mathbf{e}}_0 & \bar{\mathbf{e}}_1 \end{pmatrix} \begin{pmatrix} z_1^{-1} & 0 \\ az_1^{-1}z_2^{-1} & z_2^{-1} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{b}}_0^T \\ \bar{\mathbf{b}}_1^T \end{pmatrix} \quad (4.45a)$$

and

$$\mathbf{G}_p(z_1, z_2) = \mathbf{D}^{-T} - \mathbf{D}^{-T} \begin{pmatrix} \bar{\mathbf{b}}_0 & \bar{\mathbf{b}}_1 \end{pmatrix} \begin{pmatrix} z_1^{-1} & (a-b)z_1^{-1}z_2^{-1} \\ 0 & z_2^{-1} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{e}}_0^T \\ \bar{\mathbf{e}}_1^T \end{pmatrix}, \quad (4.45b)$$

where

$$\bar{\mathbf{b}}_0 \perp \bar{\mathbf{e}}_0, \bar{\mathbf{b}}_0 \perp \bar{\mathbf{e}}_1, \bar{\mathbf{b}}_1 \perp \bar{\mathbf{e}}_1 \text{ and } \bar{\mathbf{b}}_1^T \bar{\mathbf{e}}_0 = b.$$

Given Eq. (4.43b), we can determine the number of free variables we have for a FIR system whose inverse is also FIR. We get that  $\mathbf{B}$  has  $Nm - \frac{m(m+1)}{2}$  free variables ( $\bar{\mathbf{b}}_0$  has  $N - m$  free variables,  $\bar{\mathbf{b}}_1$  has  $N - m + 1$  and so on).  $\mathbf{E}$ , which is unconstrained, has  $Nm$  free variables,  $\mathbf{D}$  has  $N^2 - 1$  (one constraint to force non-singularity), and finally,  $\mathbf{A}$  has  $\frac{m(m-1)}{2}$  free variables (its first column has  $m - 1$ , the second  $m - 2$  and so forth). The total number is therefore

$$F_{np} = N^2 + (2N - 1)m - 1 = N^2 + (2N - 1)(m_1 + m_2) - 1. \quad (4.46)$$

If we compare  $F_{np}$  with the number of unconstrained variables of the paraunitary system (see Eq. (4.38)) then  $F_{np} - F_p = \frac{1}{2}N(N + 2m + 1) - 1$ . Clearly, the non-paraunitary system has considerably more free variables which is an indication of how well a specific design goal can be met.

The only limitation with the described design method is that it presumes that  $\mathbf{D}$  is non-singular. It is not clear what this condition implies in terms of restricted filter properties. For example, the  $\mathbf{D}$  is singular for paraunitary systems; the only

requirement is that  $\mathbf{D}^T \mathbf{D} + \mathbf{B}^T \mathbf{B} = \mathbf{I}$ . It is worth noting that the constant terms of the transfer function come from  $\mathbf{D}$ . These terms correspond to the filter coefficients within the first unit cell (*i.e.*,  $h_i(n_1, n_2)$  for  $0 \leq n_1 < d_{00}$  and  $0 \leq n_2 < d_{11}$ ,  $\forall i \in (0, N]$ ).

#### 4.6.2 Discussion of design based on the Smith form

The polyphase matrix of a system can be written in its Smith form [MOR77]:

$$\mathbf{H}_p(z_1, z_2) = \mathbf{R}(z_1, z_2) \mathbf{\Lambda}(z_1, z_2) \mathbf{S}(z_1, z_2). \quad (4.47)$$

$\mathbf{R}(z_1, z_2)$  and  $\mathbf{S}(z_1, z_2)$  are matrices with determinants as function of  $z_2$  only, namely  $\mathcal{R}(z_2)$  and  $\mathcal{S}(z_2)$ , respectively (*i.e.*,  $\mathbf{R}$  and  $\mathbf{S}$  are unimodular in  $z_1$ ). The Smith form is  $\mathbf{\Lambda}(z_1, z_2) = \text{diag}(\lambda_0(z_1, z_2), \dots, \lambda_{N-1}(z_1, z_2))$ , where the polynomials  $\lambda_i$  are monic and such that  $\lambda_i$  divides  $\lambda_{i+1}$ . For a given matrix,  $\mathbf{H}_p$ , only the Smith form  $\mathbf{\Lambda}$  is unique; different  $\mathbf{R}$ 's and  $\mathbf{S}$ 's may yield the same  $\mathbf{H}_p$ .

For any perfect reconstruction system with FIR filters in both the analysis and the synthesis filter banks, the determinant of  $\mathbf{H}_p(z_1, z_2)$  is a monomial in  $z_1$  and  $z_2$ . If we call the determinant of the Smith form  $\mathcal{L}(z_1, z_2)$  then

$$\det(\mathbf{H}_p(z_1, z_2)) = \mathcal{R}(z_2) \mathcal{L}(z_1, z_2) \mathcal{S}(z_2) = c z_1^{-m_1} z_2^{-m_2}. \quad (4.48)$$

This implies that the determinants of  $\mathbf{R}$ ,  $\mathbf{S}$  and  $\mathbf{\Lambda}$  are all monomials. In particular, it implies that all the  $\lambda_i$ 's are monomials and, thus, the Smith form is paraunitary. Consequently, for every non-paraunitary perfect reconstruction FIR system, there exists a Smith-form equivalent paraunitary system. If both  $\mathbf{R}(z_1, z_2)$  and  $\mathbf{S}(z_1, z_2)$  are paraunitary, so is  $\mathbf{H}_p(z_1, z_2)$ .

In analogy with the one-dimensional case, we propose that  $\mathbf{R}(z_1, z_2)$  and  $\mathbf{S}(z_1, z_2)$  can be designed as products of finite numbers of elementary matrices, each

corresponding to an elementary row or column operation (see [KAI80]). There are three types of elementary row and column operations, as listed below.

1. Multiply a row by a polynomial in  $z_2$  (*i.e.*,  $\mathbf{R} = \text{diag} (1, \dots, 1, P(z_2), 1, \dots, 1)$ ).
2. Permute two rows ( $\mathbf{R}$  is constructed from an identity matrix by exchanging rows  $k$  and  $l$ ).
3. Add one row, multiplied by an arbitrary polynomial, to any other row (*i.e.*,  $\mathbf{R}(z_1, z_2) = \mathbf{I} + Q(z_1, z_2)_{i,j}$ , where  $i, j$  is the row and column location of the polynomial, and  $i \neq j$ ).

Elementary matrices of type 2 and 3 have determinants equal to unity, and only type 1 has a determinant as a function of  $z_2$ . Since the determinant is limited to be a monomial,  $P(z_2) = a z_2^{-1}$  ( $a$  is an arbitrary constant). The inverse of the unimodular matrices would then be given by a cascade of the inverse elementary matrices.

Lastly, we wish to point out that we do not know if the three elementary matrix types are sufficient to construct all two-dimensional matrices which have a one-dimensional determinant. (Smith-form based design is treated in [VIS88] for scalar matrices  $\mathbf{R}$  and  $\mathbf{S}$ .)

#### 4.7 Linear phase conditions for FIR filter banks

When images are subband coded it is often desirable to have filters with linear phase response. With non-linear phase response, coding loss can result in phase error in addition to amplitude error. However, phase error is usually regarded as more visible in images compared to amplitude error. It should therefore be avoided by use of linear phase filters. In what follows, conditions will be given to test filter banks for linear phase in the polyphase domain, as given by the previously discussed

design procedure.

Assume that  $h_i(n_1, n_2)$  is a coefficient in the two-dimensional impulse response of filter  $i$  and that the filter has finite size  $M_1 \times M_2$ , as specified on the input lattice. If the filter has linear phase response, the coefficients have the following symmetry:

$$h_i(n_1, n_2) = \pm h_i(M_1 - n_1 - 1, M_2 - n_2 - 1). \quad (4.49)$$

The possible sign shift appears if the impulse response is antisymmetric. The sign is thus either  $+$  or  $-$  for the entire filter. This condition can also be written as

$$\mathbf{H}_i = \pm \mathbf{J} \mathbf{H}_i \mathbf{J}, \quad (4.50)$$

where  $\mathbf{J}$  is a matrix with unit elements along the antidiagonal and

$$\mathbf{H}_i = \begin{pmatrix} h_i(0, 0) & \cdots & h_i(0, M_2 - 1) \\ \vdots & \ddots & \vdots \\ h_i(M_1 - 1, 0) & \cdots & h_i(M_1 - 1, M_2 - 1) \end{pmatrix}. \quad (4.51)$$

A central symmetry is thus necessary for linear phase. A matrix which meets the condition of Eq. (4.50) is referred to as *persymmetric* or *anti-persymmetric* for the ‘ $+$ ’ and ‘ $-$ ’, respectively. The condition, as given in Eq. (4.51), is easily translated into the  $z$ -transform domain:

$$\begin{aligned} H_i(z_1, z_2) &= (1 \quad z_1^{-1} \quad \cdots \quad z_1^{-M_1+1}) \mathbf{H}_i \begin{pmatrix} 1 \\ z_2^{-1} \\ \vdots \\ z_2^{-M_2+1} \end{pmatrix} \\ &= \pm (1 \quad \cdots \quad z_1^{-M_1+1}) \mathbf{J} \mathbf{H}_i \mathbf{J} \begin{pmatrix} 1 \\ \vdots \\ z_2^{-M_2+1} \end{pmatrix} \\ &= \pm z_1^{-M_1+1} z_2^{-M_2+1} H_i(z_1^{-1}, z_2^{-1}). \end{aligned} \quad (4.52)$$

The linear phase condition of Eq. (4.49) is in fact valid for any shape of the impulse response. In this general case,  $M_1$  and  $M_2$  are the dimensions of the parallelogram

which bounds the impulse response on the input lattice, and the points outside the impulse response are taken to be zero.

The aim is to see how the condition in Eq. (4.49) translates for the polyphase representation of the filter bank. For a coefficient  $h_i(n_1, n_2)$ , contained in polyphase component  $(k, l)$ , the corresponding coefficient  $h_i(M_1 - n_1 - 1, M_2 - n_2 - 1)$  is contained in polyphase component  $(k', l')$ . It can be shown that the two are related by

$$\begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} M_1 - k - 1 \\ M_2 - l - 1 \end{pmatrix} - \mathbf{D} \begin{pmatrix} \left\lfloor \frac{M_1 - k - 1}{d_{00}} \right\rfloor \\ \left\lfloor \frac{M_2 - l - 1}{d_{11}} \right\rfloor \end{pmatrix}, \quad (4.53)$$

where  $\mathbf{D}$  is the subsampling matrix of Eq. (4.1), and  $\lfloor x \rfloor$  gives the integer part of  $x$  (i.e., a floor-function). The polyphase index  $k'$  should be evaluated *modulo*  $d_{00}$  and the index  $l'$  *modulo*  $d_{11}$ . In analogy with Eq. (4.52), the linear phase condition for polyphase component  $(k, l)$  becomes

$$H_{pi,k,l}(z_1, z_2) = \pm z_1^{-(u_{1max} + u_{1min})} z_2^{-(u_{2max} + u_{2min})} H_{pi,k',l'}(z_1^{-1}, z_2^{-1}). \quad (4.54)$$

Here, we denote the highest and lowest power of  $z_1$  with  $u_{1max}$  and  $u_{1min}$ , respectively, and analogously for  $z_2$ .

For a finite impulse response defined on the input lattice, the polyphase components are computed as

$$H_{pi,k,l}(z_1, z_2) = \sum_{u_2=0}^{\left\lfloor \frac{M_2-l-1}{d_{11}} \right\rfloor} \sum_{u_1=-\left\lfloor \frac{u_2 d_{01} + k}{d_{00}} \right\rfloor}^{\left\lfloor \frac{M_1-k-1-u_2 d_{01}}{d_{00}} \right\rfloor} h_i(k + d_{00}u_1 + d_{01}u_2, l + d_{11}u_2) z_1^{-u_1} z_2^{-u_2}, \quad (4.55)$$

where  $k = 0, \dots, d_{00} - 1$  and  $l = 0, \dots, d_{11} - 1$  (confer Eq. (4.9)). (Note that although some polyphase components are not causal in  $z_1$ , it is irrelevant as long as the impulse response is causal.) The highest and lowest powers in  $z_1$  and  $z_2$  of a

polyphase component,  $(k, l)$ , are given by

$$u_{1min} = - \left\lfloor \frac{u_{2max} d_{01} + k}{d_{00}} \right\rfloor \quad \text{and} \quad u_{1max} = \left\lfloor \frac{M_1 - k - 1}{d_{00}} \right\rfloor; \quad (4.56a)$$

$$u_{2min} = 0 \quad \text{and} \quad u_{2max} = \left\lfloor \frac{M_2 - l - 1}{d_{11}} \right\rfloor. \quad (4.56b)$$

However, this makes the condition in Eq. (4.54) intractable since  $u_{1min}$  is dependent on  $u_{2max}$  and since they depend on  $k$  and  $l$ . By imposing restrictions on the filter dimensions,  $M_1$  and  $M_2$ , the powers given in Eqs. (4.56) can be made equal for all polyphase components of a filter. A suggested choice is

$$M_1 = (m_1 + 1)d_{00} \quad \text{and} \quad M_2 = (m_2 d_{00} + 1)d_{11}. \quad (4.57)$$

This choice gives  $u_{1min} = -m_2 d_{01}$ ,  $u_{1max} = m_1$ ,  $u_{2min} = 0$ , and  $u_{2max} = m_2 d_{00}$ .

In addition for these filter dimensions, Eq. (4.53) is reduced to

$$k' = d_{00} - k - 1 \quad \text{and} \quad l' = d_{11} - l - 1. \quad (4.58)$$

Eqs. (4.54) and (4.58) are sufficient for formulating the linear phase condition for the polyphase form of the entire filter bank (given by Eq. (4.13)). For the filter dimensions of Eq. (4.57), this condition is

$$\mathbf{H}_p(z_1, z_2) = z_1^{-m_1 - m_2 d_{01}} z_2^{-m_2 d_{00}} \text{diag}[\pm 1, \dots, \pm 1] \mathbf{H}_p(z_1^{-1}, z_2^{-1}) \mathbf{J}. \quad (4.59)$$

In the equation above, the diagonal matrix has value  $-1$  in row  $i$  if the  $i$ th filter has odd symmetry (as given by Eq. (4.49)) else the value is  $+1$ .  $\mathbf{J}$  corresponds to the reordering of the components as given by Eq. (4.58).

Alternatively, the filter size can be specified in terms of unit-cells, say  $w_1$  along  $u_1$  and  $w_2$  along  $u_2$  on the subsampling lattice. This is in fact more likely to be the case with the design procedures previously described since they aim at constructing

the polyphase representation of a filter bank. In this case, polyphase component  $(k, l)$  is defined as

$$H_{pi,k,l}(z_1, z_2) = \sum_{u_2=0}^{w_2-1} \sum_{u_1=0}^{w_1-1} h_i(k + d_{00}u_1 + d_{01}u_2, l + d_{11}u_2) z_1^{-u_1} z_2^{-u_2}. \quad (4.60)$$

This makes the region of support for the impulse response bounded on the input lattice by  $M_1 = w_1 d_{00} + (w_2 - 1)d_{01}$  and  $M_2 = w_2 d_{11}$ . Again, the relation between polyphase components  $(k', l')$  and  $(k, l)$  is given by Eq. (4.58). The linear phase condition for an entire filter bank is

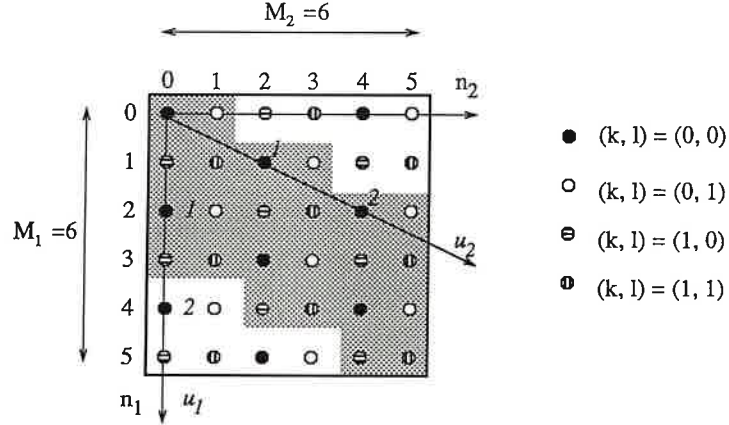
$$\mathbf{H}_p(z_1, z_2) = z_1^{-(w_1-1)} z_2^{-(w_2-1)} \text{diag}[\pm 1, \dots, \pm 1] \mathbf{H}_p(z_1^{-1}, z_2^{-1}) \mathbf{J}, \quad (4.61)$$

where the various factors have the same meaning as for Eq. (4.59).

*Example 6 For a hexagonal subsampling, choose  $m_1 = 2$  and  $m_2 = 1$  in Eq. (4.57), which gives  $M_1 \times M_2 = 6 \times 6$  and  $u_{1\min} = -1$ ,  $u_{1\max} = 2$  and  $u_{2\max} = 2$ . From Eq. (4.58) we get  $k' = 1 - k$  and  $l' = 1 - l$ . If we assume that the first two filters are symmetric and the other two are anti-symmetric, the condition of Eq. (4.59) becomes:*

$$\begin{aligned} \mathbf{H}_p(z_1, z_2) &= z_1^{-1} z_2^{-2} \text{diag}(1 \quad 1 \quad -1 \quad -1) \mathbf{H}_p(z_1^{-1}, z_2^{-1}) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} H_{p0,0,0}(z_1, z_2) & \cdots & H_{p0,1,1}(z_1, z_2) \\ \vdots & \vdots & \vdots \\ H_{p3,0,0}(z_1, z_2) & \cdots & H_{p3,1,1}(z_1, z_2) \end{pmatrix} \\ &= z_1^{-1} z_2^{-2} \begin{pmatrix} H_{p0,1,1}(z_1^{-1}, z_2^{-1}) & \cdots & H_{p0,0,0}(z_1^{-1}, z_2^{-1}) \\ \vdots & \vdots & \vdots \\ -H_{p3,1,1}(z_1^{-1}, z_2^{-1}) & \cdots & -H_{p3,0,0}(z_1^{-1}, z_2^{-1}) \end{pmatrix}. \end{aligned}$$

This filter is illustrated in Fig. 4.5. In the figure the region of support is marked for a filter with  $w_1 = 2$  and  $w_2 = 3$ .



**Figure 4.5** The polyphase components for hexagonal subsampling of a  $6 \times 6$  filter. The shaded area marks the region of support for a filter of size  $2 \times 3$  unit cells.

#### 4.8 Design of linear phase systems

Aided by the linear phase condition of the previous section, it is possible to derive design structures for linear phase systems.

Assume a linear phase perfect reconstruction system,  $\mathbf{H}_p(z_1, z_2)$ , with filter dimensions given by Eq. (4.60) and which meets the condition of Eq. (4.61). In order to iteratively create filter banks of larger size, the matrix  $\mathbf{H}_p(z_1, z_2)$  is multiplied by the “extension”  $\mathbf{E}_p(z_1, z_2)$ . The new system should remain a perfect reconstruction one with linear phase. The filters of the new system are bounded by  $M'_1 \times M'_2$  which will be set to  $\mu_1 d_{00} + (\mu_2 - 1)d_{01} \times \mu_2 d_{11}$ , and where  $\mu_1 \geq w_1$  and  $\mu_2 \geq w_2$ . When the linear phase condition is imposed, we get

$$\begin{aligned} \mathbf{H}_p(z_1, z_2) \mathbf{E}_p(z_1, z_2) &= z_1^{-(\mu_1-1)} z_2^{-(\mu_2-1)} \text{diag}[\pm 1, \dots, \pm 1] \\ &\quad \times \mathbf{H}_p(z_1^{-1}, z_2^{-1}) \mathbf{E}_p(z_1^{-1}, z_2^{-1}) \mathbf{J}. \end{aligned} \quad (4.62)$$

It is assumed that the symmetries of the filters are not affected by the extension (*i.e.*, the diagonal matrix of  $\pm 1$ 's is unaffected). Then, by comparing Eq. (4.62) with Eq. (4.61), the following condition on  $\mathbf{E}_p(z_1, z_2)$  falls out

$$\mathbf{E}_p(z_1, z_2) = z_1^{-(\mu_1-w_1)} z_2^{-(\mu_2-w_2)} \mathbf{J} \mathbf{E}_p(z_1^{-1}, z_2^{-1}) \mathbf{J}. \quad (4.63)$$

We assume that  $\mathbf{E}_p(z_1, z_2)$  can be designed as

$$\mathbf{E}_p(z_1, z_2) = \mathbf{\Delta}(z_1, z_2) \mathbf{E}, \quad (4.64)$$

where  $\mathbf{\Delta}$  is a diagonal matrix with monic monomials as elements and  $\mathbf{E}$  is a scalar matrix. The condition in Eq. (4.63) then leads to

$$\mathbf{E} = \mathbf{J} \mathbf{E} \mathbf{J} \quad \Rightarrow \quad \mathbf{E} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{E}_1 \\ \mathbf{J} \mathbf{E}_1 \mathbf{J} & \mathbf{J} \mathbf{E}_0 \mathbf{J} \end{pmatrix}. \quad (4.65)$$

A matrix that fulfills this condition is referred to as *persymmetric*, for which the elements obey  $e_{ij} = e_{(N-i)(N-j)}$ . Furthermore, we get

$$\mathbf{\Delta}(z_1, z_2) = z_1^{-(\mu_1 - w_1)} z_2^{-(\mu_2 - w_2)} \mathbf{J} \mathbf{\Delta}(z_1^{-1}, z_2^{-1}) \mathbf{J}. \quad (4.66)$$

It is easy to see that for an element  $\delta_j$  in  $\mathbf{\Delta}(z_1, z_2)$  it must meet the following condition:

$$\delta_j(z_1, z_2) = z_1^{-(\mu_1 - w_1)} z_2^{-(\mu_2 - w_2)} \delta_{N-j}(z_1^{-1}, z_2^{-1}). \quad (4.67)$$

It is therefore possible to iteratively design linear phase filter banks which give perfect reconstruction by the structure

$$\mathbf{H}'_p(z_1, z_2) = \mathbf{H}_p(z_1, z_2) \mathbf{\Delta}(z_1, z_2) \mathbf{E}. \quad (4.68)$$

where  $\mathbf{E}$  is persymmetric, and  $\mathbf{\Delta}(z_1, z_2)$  is a diagonal matrix of delays which meet Eq. (4.67).<sup>\*</sup> When  $\mathbf{H}_p$  is paraunitary and  $\mathbf{E}$  is orthogonal, the system is paraunitary, else non-paraunitary. For instance, a low order system could be based on  $\mathbf{H}_p$  being a scalar matrix, like the Walsh-Haddamard transform. In the paraunitary case, the inverse system is given by

$$\mathbf{G}'_p(z_1, z_2) = F(z_1, z_2) \mathbf{H}_p(z_1^{-1}, z_2^{-1}) \mathbf{\Delta}(z_1^{-1}, z_2^{-1}) \mathbf{E}, \quad (4.69)$$

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<sup>\*</sup> This form was suggested by Jelena Kovačević for [KAR88c].

where the monic monomial  $F(z_1, z_2)$  should be chosen so that the synthesis system is causal. For an invertible system, the synthesis is given by

$$\mathbf{G}'_p(z_1, z_2) = \mathbf{H}_p(z_1, z_2)^{-T} \mathbf{\Delta}(z_1^{-1}, z_2^{-1}) \mathbf{E}^{-T}, \quad (4.70)$$

provided that  $\mathbf{E}$  is non-singular (superscript ‘ $-T$ ’ denote the transposed inverse). It is our belief that the suggested structures may yield useful linear phase filter banks.

*Example 7* For a four band system, the design would be achieved by matrices of the forms

$$\mathbf{E} = \begin{pmatrix} e_{00} & e_{01} & e_{02} & e_{03} \\ e_{10} & e_{11} & e_{12} & e_{13} \\ e_{13} & e_{12} & e_{11} & e_{10} \\ e_{03} & e_{02} & e_{01} & e_{00} \end{pmatrix}$$

and

$$\mathbf{\Delta}(z_1, z_2) = \begin{pmatrix} z_1^{-\alpha} z_2^{-\beta} & 0 & 0 & 0 \\ 0 & z_1^{-\gamma} z_2^{-\delta} & 0 & 0 \\ 0 & 0 & z_1^{\gamma-\zeta_1} z_2^{\delta-\zeta_2} & 0 \\ 0 & 0 & 0 & z_1^{\alpha-\zeta_1} z_2^{\beta-\zeta_2} \end{pmatrix},$$

where  $\zeta_1 = \mu_1 - w_1$  and  $\zeta_2 = \mu_2 - w_2$ . The powers  $\alpha$  and  $\gamma$  can be in the range  $0 \cdots (\mu_1 - w_1)$ , and  $\beta$  and  $\delta$  can be in the range  $0 \cdots (\mu_2 - w_2)$ .

A specific design example of a non-paraunitary linear phase filter bank is given below for  $w_1 = w_2 = 0$  and  $\mu_1 = \mu_2 = 1$ :

$$\mathbf{H}_p(z_1, z_2) = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}}_{\mathbf{W}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1^{-1} & 0 & 0 \\ 0 & 0 & z_2^{-1} & 0 \\ 0 & 0 & 0 & z_1^{-1} z_2^{-1} \end{pmatrix}}_{\mathbf{\Delta}} \underbrace{\begin{pmatrix} 1 & a_2 & a_1 & a_3 \\ a_2 & 1 & a_3 & a_1 \\ a_1 & a_3 & 1 & a_2 \\ a_3 & a_1 & a_2 & 1 \end{pmatrix}}_{\mathbf{E}}.$$

As given by Eq. (4.11), the polyphase components of the first filter become

$$\mathbf{H}_{p0}(z_1, z_2) = \begin{pmatrix} 1 + a_2 z_1^{-1} + a_1 z_2^{-1} + a_3 z_1^{-1} z_2^{-1} & a_1 + a_3 z_1^{-1} + z_2^{-1} + a_2 z_1^{-1} z_2^{-1} \\ a_2 + z_1^{-1} + a_3 z_2^{-1} + a_1 z_1^{-1} z_2^{-1} & a_3 + a_1 z_1^{-1} + a_2 z_2^{-1} + z_1^{-1} z_2^{-1} \end{pmatrix}.$$

The other three filters have polyphase components which are similar; only sign changes differ the polynomials. If we evaluate the polyphase matrix for a separable subsampling pattern  $2 \times 2$ , then we get the filters

$$\begin{aligned} \mathbf{H}_0 &= \begin{pmatrix} 1 & a_1 & a_1 & 1 \\ a_2 & a_3 & a_3 & a_2 \\ a_2 & a_3 & a_3 & a_2 \\ 1 & a_1 & a_1 & 1 \end{pmatrix} & \mathbf{H}_1 &= \begin{pmatrix} 1 & a_1 & a_1 & 1 \\ a_2 & a_3 & a_3 & a_2 \\ -a_2 & -a_3 & -a_3 & -a_2 \\ -1 & -a_1 & -a_1 & -1 \end{pmatrix} \\ \mathbf{H}_2 &= \begin{pmatrix} 1 & a_1 & -a_1 & -1 \\ a_2 & a_3 & -a_3 & -a_2 \\ a_2 & a_3 & -a_3 & -a_2 \\ 1 & a_1 & -a_1 & -1 \end{pmatrix} & \mathbf{H}_3 &= \begin{pmatrix} 1 & a_1 & -a_1 & -1 \\ a_2 & a_3 & -a_3 & -a_2 \\ -a_2 & -a_3 & a_3 & a_2 \\ -1 & -a_1 & a_1 & 1 \end{pmatrix} \end{aligned}$$

Note that the matrix  $\mathbf{E}$ , which is not orthogonal, was chosen to obtain the symmetries in the filters. The corresponding synthesis filters are then given by

$$\mathbf{G}_p(z_1, z_2) = \frac{1}{4} z_1^{-1} z_2^{-1} \mathbf{W} \Delta(z_1^{-1}, z_2^{-1}) \mathbf{E}^{-T},$$

where  $\mathbf{W}$  is the Walsh-Hadamard transform.  $\mathbf{E}^{-1}$  retains the structure of  $\mathbf{E}$  so that the synthesis filters,  $\mathbf{G}_0 \dots \mathbf{G}_3$ , have the same symmetries as the analysis filters above.

#### 4.9 Summary and extensions

This chapter has presented new results on the theory of general two-dimensional multirate filter banks. The multirate theory is general with no restrictions on the geometries of the input and subsampling lattices. By decomposition of a filter bank into its polyphase components, the input/output relation of a subband analysis and synthesis system is determined leading to a necessary and sufficient condition for alias-free and perfect signal reconstruction. Then, working on the polyphase decomposition, design structures were given which gave rise to the design of two-dimensional perfect reconstruction filter banks. Conditions were derived to test

these filter banks in the polyphase domain for linear phase, which permitted us to develop a restricted design structure for linear phase systems.

Two-dimensional, non-separable, filter banks have not received much attention in the research literature. The theory therefore offers a wide range of extensions when going from one dimension to two. Of the material covered in Sections 4.1 to 4.3 the most interesting extension is to find a necessary and sufficient matrix structure for  $\mathbf{T}_p$  for any given subsampling pattern. This can lead to matrix implementations of scalar transfer functions, by polyphase decomposition of the input signal.

The whole field of filter bank design lies open in the two-dimensional case. Research could be aimed at finding a sufficient cascade-structure with proven completeness over the polyphase matrices which have the determinant as a monomial. The paraunitary condition can then be imposed on this form. A most important extension is to apply the design methods proposed herein in order to find usable filter banks. Then, finally, a comparison can be made between separable and non-separable subband coding of images.