SF2812 Applied linear optimization, final exam
Friday March 11 2022 8.00–13.00

Examiner: Jan Kronqvist, tel. 08 790 71 37.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) and its dual (DLP) be defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 3.0 \\ -3.0 \\ 5.8 \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} -4 & -4 & -2 & -1 & -1 \end{pmatrix}^T.$$

(a) A person named JK has used GAMS to model and solve this problem. JK has been told that he can solve either (LP) or (DLP) for finding the optimal solutions to (LP) and (DLP). He has chosen to solve (LP). The GAMS input file can be found at the end of the exam, and a partial GAMS output file reads:

```
                S O L V E      S U M M A R Y
     MODEL   primal_LP           OBJECTIVE  objective_var
     TYPE    LP                  DIRECTION  MINIMIZE
     SOLVER  CPLEX               FROM LINE  17

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE              -29.6000

Optimal solution found
Objective:           -29.600000

Equations:
                         LOWER         LEVEL         UPPER          MARGINAL
--- EQU constr_1         3.0000        3.0000        3.0000         -4.0000
--- EQU constr_2        -3.0000       -3.0000       -3.0000          2.0000
--- EQU constr_3         5.8000        5.8000        5.8000         -2.0000
```

```
Variables:
                          LOWER        LEVEL       UPPER        MARGINAL
--- VAR x1                  .          3.0000      +INF            .
--- VAR x2                  .          3.0000      +INF            .
--- VAR x3                  .          2.8000      +INF            .
--- VAR x4                  .            .         +INF          3.0000
--- VAR x5                  .            .         +INF          1.0000
--- VAR objective         -INF       -29.6000      +INF            .
```

The only catch is that JK is not familiar with GAMS, and does not know how to extract the optimal solutions from the GAMS output. Help JK obtain the optimal solutions to (LP) and (DLP) from the GAMS output file. ......(4p)

(b) JK claims that if b_2 is changed to $-3+\delta$ and b_3 simultaneously is changed to $5.8+\delta$, then the optimal value remains unchanged (assuming δ is small enough). Show that JK is right. Do so without solving any system of equations. ..(3p)

(c) Determine how much the objective coefficients for variables x_4 and x_5 can be increased/decreased before the optimal solution changes ....(3p)
Hint: To avoid difficult matrix operations, it might be good to remember that the dual variables y solve the system $B^T y = c_b$, i.e., $y = B^{-T} c_b$.

2. Consider a mixed-integer linear program $(MILP)$ of the form

$$(MILP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathbb{R}^n, \\ & x_i \in \{0,1\} \quad \forall i \in \mathcal{I} \subset \{1,2,\ldots,n\}. \end{array}$$

(a) Assume that we only have three binary variables defined by $\mathcal{I} = \{1,2,3\}$, and assume that (MILP) contains the constraint $x_1 + x_2 + x_3 = 1$. However, (MILP) contains a large number of continuous variables. We solve this problem by branch-and-bound with linear programming relaxations at the nodes. Show that the branch-and-bound tree will have at most five nodes. You may assume that the linear programs that arise have unique optimal solutions and that they can be solved without any issues or be infeasible. .... (5p)
Hint: Draw a branch-and-bound tree and analyze its possible shape. At which nodes can the subproblems return non-integer solutions.

(b) Describe how the number of nodes could potentially be reduced by knowing a good feasible solution before starting the branch-and-bound procedure. . (2p)

(c) Now assume $\mathcal{I} = \{1,2,3,4\}$ and that (MILP) contains the constraint

$$\frac{(x_1 + x_2 + x_3)}{3} \leq x_4,$$

but not the constraint from Exercise 2a. This constraint originates from a logical relation between the integer variables x_1, x_2, x_3 and x_4. A person named JK claims that this logical dependence between the variables can also be exactly represented by the constraints

$$x_1 \leq x_4, \quad x_2 \leq x_4, \quad x_3 \leq x_4.$$

Show that any solution (fractional or integer) satisfying the latter set of constraints will satisfy the first single constraint. Furthermore, show that using the latter set of constraints can result in a greater optimal objective function value for the continuous relaxation of problem (MILP). .... (3p)

3. Let (P) and (D) be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

For a fixed positive barrier parameter μ, consider the primal-dual nonlinear equations

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

where we in addition require $x > 0$ and $s > 0$. Here, $X = \text{diag}(x)$, $S = \text{diag}(s)$ and e is an n-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive μ, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to (P) and $y(\mu), s(\mu)$ are feasible to (D) with duality gap $n\mu$. .... (3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton's method. .... (5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of μ? .... (2p)

4. Consider the binary integer programming problem (IP) given by

$$(IP) \quad \begin{array}{ll} \text{minimize} & -2x_1 - x_2 - x_3 - 0.5x_5 \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_3 + x_4 + x_5 = 1, \\ & -x_1 - x_3 \geq -1, \\ & x_j \in \{0, 1\}, \quad j = 1, \dots, 5. \end{array}$$

Assume that the constraint $-x_1 - x_3 \geq -1$, is relaxed by Lagrangian relaxation for a nonnegative multiplier u.

(a) For $u = 1$, compute two optimal solutions to the resulting Lagrangian relaxed problem. The Lagrangian relaxed problem may be solved by any method, that need not be systematic (branch-and-bound not needed). .... (3p)

(b) Use the two optimal solutions to the Lagrangian relaxed problem computed in Exercise 4a to give two different subgradients to the dual objective function φ at $u = 1$. .... (4p)

(c) Based on the two subgradients computed in Exercise 4b, can you tell if the solution is optimal for the dual problem? Furthermore, can you tell if either of the two solutions obtained in Exercise 4a are optimal for (IP)? .... (3p)

5. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \geq c. \end{array}$$

Let the dimensions of the problem be such that A is an $m \times n$ matrix and let A_i denote the ith column of A.

For $i = 1, \ldots, n$, let $\mathcal{P}_i$ be a polytope given by $\mathcal{P}_i = \{v_i \in I\!R^m : C_i^T v_i \geq d_i\}$, for given matrices C_i of dimensions $m \times n_i$ and given vectors d_i of length n_i. Each polytope $\mathcal{P}_i$ is bounded and such that $A_i \in \mathcal{P}_i$.

The reason for introducing the sets P_i is that we are interested in solving an optimization problem which is robust against uncertainties in A, given by

$$(RP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \min_{v_i \in \mathcal{P}_i} \{v_i^T y\} \geq c_i, \quad i = 1, \ldots, n. \end{array}$$

(a) Problem (RP) looks complicated as it has a minimization function in each constraint. Use your expertise in linear programming to formulate a linear program which is equivalent to (RP). .... (6p)

Hint 1: For a given y, each problem in the constraints

$$\begin{array}{ll} \underset{v_i \in I\!R^m}{\text{minimize}} & y^T v_i \\ \text{subject to} & C_i^T v_i \geq d_i, \end{array}$$

is a linear program.

Hint 2: Use strong duality for linear programming.

(b) Derive the dual problem associated with the linear program you obtained in Exercise 5a. .... (4p)

– If you were not able to solve Exercise 5a, then you may assume the equivalent linear program is

$$\begin{array}{ll} \underset{x,y}{\text{maximize}} & f^T x \\ \text{subject to} & D_i x \geq b_i, \quad i = 1, \ldots, n, \\ & y - Bx = 0, \\ & y \geq 0, \end{array}$$

and use this in Exercise 5b. Assume the matrixes B, D_i and vectors f, b_i are of suitable size. Note, this is not the solution to Exercise 5a.

Good luck!

GAMS file for exercise 1:

```
Positive Variables x1, x2, x3, x4, x5;
Variable objective_var;

equations
constr_1,constr_2,constr_3,objective;

objective.. -4*x1 - 4*x2 -2*x3 -1*x4 -1*x5 =E= objective_var;

constr_1.. x1 + x4  =E= 3;
constr_2.. -x2 -x5  =E= -3;
constr_3.. x2 + x3  =E= 5.8;



Model primal_LP /all/;

Solve primal_LP using lp minimizing objective_var;
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