SF2812 Applied linear optimization, final exam
Friday March 11 2022 08.00–12.00
Brief solutions

1. (a) The solution to (LP) that is the x variables is the level of the variables in the GAMS output. So,

$$x = \begin{pmatrix} 3 \\ 3 \\ 2.8 \\ 0 \\ 0 \end{pmatrix},$$

with objective function value -29.6. The variables y are dual variables of the constraints $Ax = b$, and are the marignals in the GAMS output. Similarly, s are the dual variables of the constraints $x \geq 0$. From the GAMS output we get

$$y = \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}.$$

(b) With a small enough δ, the optimal basis will note change (same partitioning of basic and nonbasic variables remains optimal). Remember $x_B = B^{-1}b$. The optimal objective function value is then given by

$$c_B^T B^{-1} b = b^T B^{-T} c_B = b^T y. \qquad \text{(this derivation not needed in the exam)}$$

By strong duality you also get that the objective function value is given $b^T y$. Now, with $\tilde{b} = b + [0 \quad \delta \quad \delta]^T$ we get

$$\text{objective value} \quad = \tilde{b}^T y = b^T y + y_2\delta + y_3\delta.$$

The change in the objective function value is, therefore, given by $y_2\delta + y_3\delta = 2\delta - 2\delta$.

(c) Note that x_4, x_5 are nonbasic variables (they are zero at the optimal solution). As long as the reduced costs remains positive the optimal solution will not change. The reduced costs are given by

$$c_N - N^T B^{-T} c_B \quad \text{(using the hint we can simplify)} \quad = c_N - N^T y.$$

Now,

$$N^T y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$

So, as long as

$$c_N \geq \begin{pmatrix} -4 \\ -2 \end{pmatrix},$$

the solution will remain optimal (we can make the coefficients as large as we want but the inequality above gives a lower bound on the coefficients).

2. (a) Remember to properly motivate your answer! We draw a branch-and-bound tree and analyze where we can obtain fractional solutions and how far we may have to branch. Where ever possible, we assume we obtain a fractional solution.

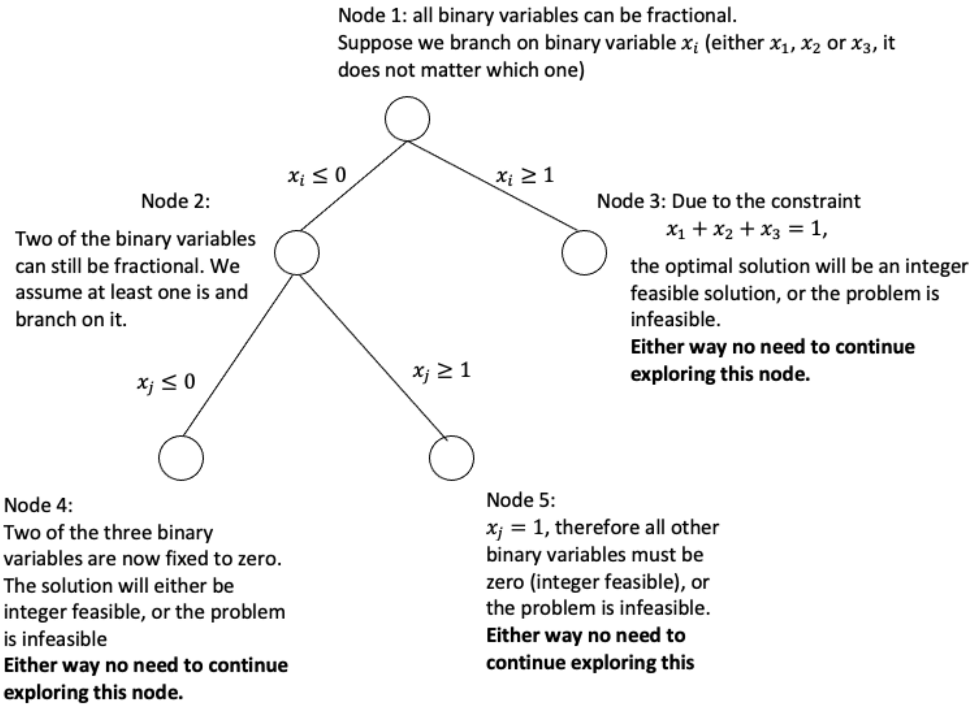

Figure 1: Possible branch-and-bound tree.

(b) If we know a good solution, then we may use this to prune nodes (stop exploring the node, even if it is fractional). The feasible solution gives us an upper bound on the optimal objective value. For example, in node 2 we can obtain a fractional solution but if the objective value is worse then the upper bound then we can fathom the node (no need to explore that node further).

(c) We can easily prove that any solution that satisfy the latter set of inequality constraints satisfy the first. By adding the inequality constraints $x_1 \leq x_4,\ x_2 \leq x_4,\ x_3 \leq x_4$ together we obtain

$$x_1 + x_2 + x_3 \leq 3x_4,$$

which is equivalent to the first constraint (holds for both integer and continuous variables).
However, any solution satisfying $\frac{x_1+x_2+x_3}{3} \leq x_4$ does not satisfy the separate constraints $x_1 \leq x_4,\ x_2 \leq x_4,\ x_3 \leq x_4$.
For example, consider $x_1 = 0.8, x_2 = 0.05, x_3 = 0.05, x_4 = 0.3$. This variable combination satisfies $\frac{x_1+x_2+x_3}{3} \leq x_4$, but clearly violates $x_1 \leq x_4$.
Therefore, using the separate constraints results in smaller feasible set. The optimal objective function value can therefore be larger when using the separate constraints. (The optimal solution obtained when minimizing with the single constraints can be excluded by the more restrictive separate constraints).

3. See Lecture note 7, or Chapter 10.2 in Griva, Nash, and Sofer (pages 321 – 324).

4. (a) for $u \geq 0$ the Lagrangian relaxed problem becomes

$$(P_u) \quad \begin{array}{ll} \text{minimize} & -2x_1 - x_2 - x_3 - 0.5x_5 - u(-x_1 - x_3 + 1) \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_3 + x_4 + x_5 = 1, \\ & x_j \in \{0,1\}, \quad j = 1, \dots, 5. \end{array}$$

For $u = 1$ we get

$$(P_1) \quad \begin{array}{ll} \text{minimize} & -x_1 - x_2 - 0.5x_5 - 1 \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_3 + x_4 + x_5 = 1, \\ & x_j \in \{0,1\}, \quad j = 1, \dots, 5. \end{array}$$

Note, there is no constraint connecting the variables x_1, x_2 with the variables x_3, x_4, x_5. We can, therefore, consider (P_1) as two independent problems

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 - 1 \\ \text{subject to} & x_1 + x_2 \leq 1, \\ & x_j \in \{0,1\} \quad j = 1, 2 \end{array} \qquad \begin{array}{ll} \text{minimize} & -0.5x_5 \\ \text{subject to} & x_3 + x_4 + x_5 = 1, \\ & x_j \in \{0,1\} \quad j = 3, 4, 5, \end{array}$$

and finding optimal solutions to these becomes trivial. The two optimal solutions are

$$x^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad x^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Remember from the lectures notes. If we have $\varphi(u) = min_{x \in X}\, c^T x - u^T(Ax - b)$, then $b - Ax(u)$ is a subgradient (where x(u) is the optimal solution to the Lagrangian relaxed problem). So the two subgradients at $u = 1$ are

$$s^1 = -1 - (-1 \cdot 1 - 1 \cdot 0) = 0 \qquad \text{and} \qquad s^2 = -1 - (-1 \cdot 0 - 1 \cdot 0) = -1$$

(c) One of the subgradients is zero. Therefore, $u = 1$ is an optimal solution to the Lagrangian relaxation (see slide 6 Lecture notes 12).

The solution x^1 that we found in exercise (a) is actually feasible for the problem (IP). If we evaluate the objective function of (IP) for x^1 we get -2.5 which is an upper bound on the optimal objective value (it is a feasible solution). Remember for any $u \geq 0$, the optimal objective value of (P_u) gives a lower bound on the optimal objective value for (IP) (this is true because it is a relaxation). With $u = 1$ we get that the optimal objective value of (P_1) is -2.5. So, we both have a lower bound on -2.5 and an upper bound of -2.5!

5. (a) This one was a bit more tricky :) First let's consider the optimization problems in the constraints

$$(LP_{C_i}) \quad \begin{array}{ll} \underset{v_i \in I\!R^m}{\text{minimize}} & y^T v_i \\ \text{subject to} & C_i^T v_i \geq d_i, \end{array}$$

where y is fixed. If we form the dual of (LP_{C_i}) we obtain

$$(DP_{C_i}) \quad \begin{array}{ll} \text{maximize} & d_i^T z_i \\ \text{subject to} & y - C_i z_i = 0, \\ & z_i \geq 0, \end{array}$$

where z_i is the dual variables.

For any z_i that satisfies the constraints in (DP_{C_i}), we know that

$$d_i^T z_i \leq Optval(LP_{C_i}).$$

This follows directly by weak duality.

We can then use this in the constraint $\min_{v_i \in \mathcal{P}_i}\{v_i^T y\} \geq c_i$ as

$$\begin{array}{l} d_i^T z_i \geq c_i \\ y - C_i z_i = 0, \\ z_i \geq 0. \end{array}$$

Note that the the constraints above could potentially be more restrictive than the original constraints, since $d_i^T z_i \leq Optval(LP_{C_i})$.

But, fortunately we have strong duality! Meaning that there exists a z_i satisfying the constraints in (DP_{C_i}) such that $d_i^T z_i = Optval(LP_{C_i})$. Therefore, the constraints

$$\begin{array}{l} d_i^T z_i \geq c_i \\ y - C_i z_i = 0, \\ z_i \geq 0. \end{array}$$

exactly represents the constraint $\min_{v_i \in \mathcal{P}_i}\{v_i^T y\} \geq c_i$.

A linear programming equivalent formulation of (RP) is, thus, given by

$$(LRP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & d_i^T z_i \geq c_i \quad i = 1, \ldots, n, \\ & y - C_i z_i = 0 \quad i = 1, \ldots, n, \\ & z_i \geq 0 \quad i = 1, \ldots, n. \end{array}$$

Note that $y, z_1, \ldots, z_n$ are all variables.

(b) Note that here it is possible to get slightly different dual problems depending on how the Lagrangian relaxation is defined. We introduce dual variables $\lambda_i \in I\!R$ for the constraints $(d_i^T z_i - c_i \geq 0)$ and dual variables $\mu_i \in I\!R^m$ for the constraints $y - C_i z_i = 0$

Next, we define the Lagrangian dual function as

$$\varphi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) = \underset{y, z_1 \geq 0, \ldots,\ z_n \geq 0}{\text{maximize}}\ b^T y + \sum_{i=1}^{n} \lambda_i (d_i^T z_i - c_i) + \sum_{i=1}^{n} \mu_i^T (y - C_i z_i).$$

By rearranging the terms we get

$$\varphi(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n) = \underset{y, z_1\geq 0,\ldots,\ z_n\geq 0}{\text{maximize}} \left(b+\sum_{i=1}^n \mu_i\right)^T y + \sum_{i=1}^n (\lambda_i d_i - C_i^T \mu_i)^T z_i - \sum_{i=1}^n \lambda_i c_i.$$

We then get

$$\varphi(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n) = \begin{cases} -\sum_{i=1}^n \lambda_i c_i & \text{if } b+\sum_{i=1}^n \mu_i = 0,\ \lambda_i d_i - C_i^T\mu_i \leq 0,\ i=1,\ldots,n, \\ \infty & \text{otherwise.} \end{cases}$$

The dual problem can then be written as

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \lambda_i c_i \\ \text{subject to} & b+\sum_{i=1}^n \mu_i = 0, \\ & \lambda_i d_i - C_i^T \mu_i \leq 0,\ i=1,\ldots,n, \\ & \lambda_i \geq 0,\ i=1,\ldots,n, \\ & \mu_i \in I\!R^m,\ i=1,\ldots,n. \end{array}$$