SF2812 Applied linear optimization, final exam
Friday March 10 2023 08.00–13.00

Examiner: Jan Kronqvist, tel. 08 790 71 37.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what has been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) and its dual (DLP) be defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \end{pmatrix}^T.$$

Your teacher JK has modeled the primal problem in GAMS and has solved it using GAMS. The GAMS model can be found at the end of the exam.

(a) JK has tested different ideas and is no longer sure which GAMS output had the optimal solution to this problem. Two GAMS outputs are given on the next page, determine which of the outputs contains the optimal solution (properly motivate your answer), and write down the optimal primal variables x and dual variables y, s .
You must motivate why the solution is optimal based on theory from the course (you can refer to theorems, but you do not need to prove any theorem for this exercise). . . . . (5p)

(b) JK is not sure if the objective coefficients are correctly chosen. Analyze how each objective coefficient in c can be increased/decreased (only increase/decrease one at a time) before the optimal x variable values change. . . . .(4p)
At some stage of the calculations, you may find it useful to know that

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ 0.4 & 0.2 & 1 \end{pmatrix}.$$

(c) If you were allowed to change one value in the vector b to improve the optimal solution, which one would you change and would you increase it or decrease it? Motivate the answer, but no calculations are needed. . . . .(1p)

GAMS output 1

```
            S O L V E      S U M M A R Y
     MODEL   LP_model           OBJECTIVE  objvar
     TYPE    LP                 DIRECTION  MINIMIZE
     SOLVER  CPLEX              FROM LINE  23

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE               -6.0000

Optimal solution found
Objective:           -6.000000

                    LOWER          LEVEL         UPPER      MARGINAL
--- EQU constr1      6.0000         6.0000         6.0000      -1.0000
--- EQU constr2      6.0000         6.0000         6.0000          .
--- EQU constr3     -1.0000        -1.0000        -1.0000          .

                    LOWER          LEVEL           UPPER          MARGINAL
---- VAR x1           .             3.0000          +INF              .
---- VAR x2           .              .              +INF            0.5000
---- VAR x3           .              .              +INF            1.0000
---- VAR x4           .             3.0000          +INF              .
---- VAR x5           .             2.0000          +INF              .
```

GAMS output 2

```
            S O L V E      S U M M A R Y
     MODEL   LP_model           OBJECTIVE  objvar
     TYPE    LP                 DIRECTION  MINIMIZE
     SOLVER  CPLEX              FROM LINE  23

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE               -3.6000

Optimal solution found
Objective:           -3.600000

                     LOWER          LEVEL          UPPER         MARGINAL
---- EQU constr1     6.0000         6.0000         6.0000         -0.4000
---- EQU constr2     6.0000         6.0000         6.0000         -0.2000
---- EQU constr3    -1.0000        -1.0000        -1.0000           .

                     LOWER          LEVEL          UPPER         MARGINAL
---- VAR x1            .            2.4000         +INF              .
---- VAR x2            .            1.2000         +INF              .
---- VAR x3            .             .             +INF            0.4000
---- VAR x4            .             .             +INF            0.2000
---- VAR x5            .            2.6000         +INF              .
```

2. Consider a linear program (LP)

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}^T.$$

(a) Start by setting x_3 and x_4 as basic variables and perform one iteration with primal simplex. Is the solution you obtained optimal? ....(6p)
At some stage of the calculations, you may find it useful to know that
$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0.5 \\ 1 & -0.5 \end{pmatrix}.$

(b) The dual of problem (LP) problem is given by

$$(DLP) \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

Using the knowledge you obtained from solving problem (LP), determine the optimal dual variables y and s. .... (4p)

3. Consider the stochastic program (P) given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \ldots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \ldots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

4. Consider the integer programming problem (IP) given by

$$(IP)\qquad \begin{array}{ll} \text{minimize} & x_1 - 2x_2 - 3x_3 - x_5 \\ \text{subject to} & x_1 + x_2 + x_3 \geq 1, \\ & x_2 + x_3 \leq 1, \\ & x_4 + x_5 + x_6 = 1, \\ & x_2 + x_4 = 1, \\ & x_j \in \{0,1\}, \quad j = 1,\dots,6. \end{array}$$

Assume that the constraint $x_2 + x_4 = 1$ is relaxed by Lagrangian relaxation with the multiplier u. (This is done by adding $-u(x_2 + x_4 - 1)$ to the objective)

(a) For $u = 2$, compute an optimal solution to the resulting Lagrangian relaxed problem. The Lagrangian relaxed problem may be solved by any method or suitable approach, and no systematic method is needed (you do not need to use branch-and-bound). .... (2p)

(b) Use an optimal solution to the Lagrangian relaxed problem computed in Exercise 4a to determine a subgradient to the objective function φ at $u = 2$ in the Lagrangian dual problem. .... (3p)

(c) Use the subgradient to update u according to the subgradient method using the steplength = 1, and determine the optimal solution (or multiple solutions) to the Lagragian relaxed problem using the updated u. .... (2p)

(d) Using the solution from Exercise 4c, can you determine if the value for u is optimal for the Lagrangian dual problem and if the optimal solution to problem (IP) has been found (motivate your answer)? .... (3p)

5. Consider a mixed-integer linear program of the form

$$(MILP)\qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & Bx \leq d, \\ & x \in \mathbb{R}^{1000}, \\ & x_i \in \{0,1\} \quad i = 1,2,3. \end{array}$$

The problem only has three binary variables (x_1, x_2, x_3) and the rest of the variables $x_4, x_5, \dots, x_{1000}$ are continuous variables. The matrixes A and B are sparse with 100 rows in A and 2500 rows in B. Furthermore, we know that (MILP) contains the constraint $2x_1 + 4x_2 + 4x_3 \leq 6$. We want to solve this problem using branch-and-bound with linear programming (LP) relaxations at the nodes.

(a) A friend of yours has read that mixed-integer problems are NP-hard and is concerned that problem ($MILP$) cannot be solved. Another person called JK claims that this problem can easily be solved using branch-and-bound, and that the maximum number of nodes that you might have to explore in the branch-and-bound tree is 13 (i.e., in the worst case you have to solve 13 LP problems). Analyze how many nodes the branch-and-bound tree can have when solving problem ($MILP$) to determine if the claim by JK is correct or not. You can assume that the linear programs that arise have unique optimal solutions and that they can be solved without any issues or be infeasible. . . . . (4p)
Hint: Draw branch-and-bound trees and analyze possible shapes. At which nodes can the subproblems return non-integer solutions.

(b) JK also claims that this problem could be solved more efficiently by adding so-called cover cuts to strengthen the continuous relaxation.
Coverer cuts are derived from constraints of the type

$$a_1y_1 + a_2y_2 + \ldots + a_ny_n \leq f,$$

where y_i are all binary variables and all $a_i > 0$. By finding an index set $I_c \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I_c} a_i > f$, a cover cut is given by the inequality

$$\text{(cover-cut)} \qquad \sum_{i \in I_c} y_i \leq |I_c| - 1,$$

where $|I_c|$ is the number of elements in the index set.

In your problem ($MILP$) you have the binary variables x_1, x_2, x_3, and your task is to derive cover cuts from the constraint $2x_1 + 4x_2 + 4x_3 \leq 6$.

i. For your problem (MILP) a cover cut can be obtained by setting $I_c = \{2, 3\}$. Motivate why this gives you a valid inequality constraint and show that adding this constraint can strengthen the LP relaxation (exclude some non-integer solutions). . . . . (2p)

ii. By choosing $I_c = \{1, 2, 3\}$ you would obtain the cover cut $x_1 + x_2 + x_3 \leq 2$. Prove that the cover cut you obtained in question (i) is stronger. By stronger, we mean that it is a stronger (more restrictive) inequality constraint. . . . . (4p)

Side note: Cover cuts are simple to obtain and can greatly improve the strength of the LP relaxation. Therefore, most (or all advanced) solver software tries to generate such cuts if possible.

Good luck!

GAMS file for exercise 1:

```
Positive Variables
x1, x2, x3, x4, x5;

Free Variables
objvar objective variable;

Equations
obj_fun the objective function
constr1 first constraint
constr2 second constraint
constr3 third constraint;

obj_fun..   -1*x1 -1*x2  =e= objvar;
constr1..   2*x1 + x2 + x3 =e= 6;
constr2..   x1 + 3*x2 + x4 =e= 6;
constr3..   -x1 -x2 + x5 =e= -1;

Model LP_model /all/;

Solve LP_model using LP minimizing objvar;
```