SF2812 Applied linear optimization, final exam
Friday March 10 2023 08.00–12.00
Brief solutions

1. (a) As you are not sure which solution is correct, you must first check if the solution is feasible. You must check that the primal variables x are feasible for (LP) and that the dual variables y, s are feasible for DLP. A solution is optimal if it is primal and dual feasible, and if $c^T x = b^T y$ (i.e., $x^T s = 0$)

By checking the constraints of the primal and dual problem, you will find that the solution given in the first GAMS output is not dual feasible.

The second GAMS output contains the solution

$$x = \begin{pmatrix} 2.4 \\ 1.2 \\ 0 \\ 0 \\ 2.6 \end{pmatrix}, \quad y = \begin{pmatrix} -0.4 \\ -0.2 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 0.4 \\ 0.2 \\ 0 \end{pmatrix},$$

and you will find that this solution is both primal and dual feasible. From the GAMS output, you can also see that the optimal objective value is -3.6. You can determine that it is optimal either from the fact that $c^T x = B^T y$ or from $x^T s = 0$.

(b) Remember the current basis remains optimal as long as the reduced costs are all positive. For the optimal solution, the variables x_1, x_2, x_5 are basic variables and x_3, x_4 nonbasic. This gives us

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now, the reduced costs are given by

$$c_N^T - c_B^T B^{-1} N \quad \text{(or as a column vector)} \quad c_N - N^T B^{-T} c_B.$$

If we start by looking at c_N (objective coefficients for x_3 and x_4), we can directly see that we can make these arbitrarily large and the solution will still remain optimal. From the reduced costs, you will get that $c_N^T \geq \begin{pmatrix} -0.4 \\ -0.2 \end{pmatrix}$ ensures that the solution remains optimal.

By analyzing the two equations defining the reduced costs, we can find the following bounds on the objective coefficients

$$-2 \leq c_1 \leq -1/3, \quad -3 \leq c_2 \leq -1/2, \quad -0.4 \leq c_3 \leq \infty, \quad -0.2 \leq c_4 \leq \infty, \quad -\infty \leq c_5 \leq 1.$$

(c) Based on the marginals, it seems most favorable to increase b_1.

2. (a) For more details on the Simplex algorithm, see Lecture 4.

- The reduced costs for the given set of basic and nonbasic variables are -2 for x_1 and 1 for x_2. Thus, you want to move x_1 into the basis.

- We can then determine the search direction $p_B = (-1 \ \ -2)^T$ and $p_N = (1 \ \ 0)^T$ and we get the maximal step length $\alpha_{max} = 2$, giving us the solution

$$x = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

The nonbasic variables are now x_2, x_4, and the reduced costs are 2 and 1 (showing that the solution is optimal).

(b) If you remembered the formulas for the Simplex algorithm on page 13 Lecture note 4 and used these, then you have already calculated the dual variables. If you did not, you can easily calculate them. If you have the optimal primal solution, then it is easy to determine the optimal dual solution and vice versa.

From the optimal x variables, you can directly determine that $s_1 = 0$ and $s_3 = 0$ (due to the complementary slackness condition, i.e., $x^T s = 0$). You can then obtain the remaining dual variables from the equality constraints of (DLP), giving you the solution

$$y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

3. See material from Lecture 6, and the supplementary course material "Introduction to Stochastic Programming, by J. R. Birge and F. Louveaux".

4. (a) The Lagrangian relaxed problem is given by

$$\begin{array}{ll} \text{minimize} & x_1 - 2x_2 - 3x_3 - x_5 - u(x_2 + x_4 - 1) \\ \text{subject to} & x_1 + x_2 + x_3 \geq 1, \\ & x_2 + x_3 \leq 1, \\ & x_4 + x_5 + x_6 = 1, \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, 6. \end{array}$$

Note that this problem can be split up into the following two independent problems

$$(P1) \begin{array}{ll} \text{minimize} & x_1 - (2+u)x_2 - 3x_3 + u \\ \text{subject to} & x_1 + x_2 + x_3 \geq 1, \\ & x_2 + x_3 \leq 1, \\ & x_j \in \{0,1\}, \quad j = 1,2,3, \end{array} \qquad (P2) \begin{array}{ll} \text{minimize} & -x_5 - ux_4 \\ \text{subject to} & x_4 + x_5 + x_6 = 1, \\ & x_j \in \{0,1\}, \quad j = 4,5,6. \end{array}$$

For these two small problems, you can easily determine optimal solutions (you can even test all possible solutions). With $u = 2$, you get the unique optimal solution

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and the objective function value -4. Therefore, you also know that -4 is a lower bound on the optimal objective function value for the original problem. But, the solution you obtained does not satisfy the constraint $x_2 + x_4 = 1$.

(b) A subgradient s is simply given by $s = -(1 + 1 - 1) = -1$. See material from lecture 12 for more details.

(c) With steplength=1, the Lagrangian multiplier u is updated according to the subgradient method by $u = u + s$. Thus, giving us $u = 1$.
With $u = 1$, there is not a unique solution to the Lagrangian relaxed problem. In the first part, problem (P1), either $x_2 = 1$ or $x_3 = 1$ and all other variables are zero. Similarly for the second part, either $x_4 = 1$ or $x_5 = 1$ and all other variables are zero. Thus we get the following 4 different optimal solutions to the problem

$$x^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(d) Here you can use the 4 different solutions to calculate the subgradients

$$s^1 = -1, \quad s^2 = 0, \quad s^3 = 0, \quad s^4 = 1.$$

As zero is a subgradient of the Lagrangian dual problem, you know that u=1 is an optimal solution to the Lagrangian dual problem.
The optimal objective function value of the Lagrangian relaxed problem (and the Lagragian dual problem) is -3. Thus, you know that -3 is a lower bound on the optimal objective function value for the original problem. This alone does not tell you if either x^1, x^2, x^3 or x^4 is an optimal solution to the original problem, nor if -3 is the optimal objective function value.

If you examine the solutions x^1, x^2, x^3 and x^4 you will find that x^2 satisfies all constraints of the original problem with an objective function value of -3. Since this is a feasible solution, you now now that -3 is an upper bound on the optimal objective value. Since both the upper and lower bound is equal to -3, this proves that x^2 is an optimal solution to the original problem.
(The optimum is not unique, in fact x^3 is also an optimal solution.)

5. (a) Based on the information we have, we can only stop at a node if the solution is integer feasible or the problem becomes infeasible (cannot satisfy $2x_1 + 4x_2 + 4x_3 \leq 6$).

It is important to remember that the order in which you branch on variables can affect the number of nodes you have to explore. Here the branching order does not affect the total number of nodes, but this is not obvious, and you need to analyze the impact of the branching order for full points (or carefully motivate why it does not matter).

(Side note: one of the main challenges in a solver is to decide which variable to branch on in a node, as there are normally many possible branching options. In practice, the choice of which variable to branch on can have a great impact on the number of nodes that must be explored and the time needed to solve the problem. But, choosing the best variable to branch on is not easy, and this is an active research topic.)

To answer how many nodes you might have to explore in the worst case, we will assume that all variables will take a fractional value at all nodes whenever possible. We must also investigate if different branching order can affect the number of nodes. We can simplify the analysis if we realize that the problem is symmetric in the variables x_2 and x_3 as branching on either one of them will have the same effect.

First, let's consider the case when we start by branching on x_1. Due to the symmetry on x_2 and x_3 we can simultaneously consider the cases where we either branch on x_2 or x_3 (otherwise we will have to consider more cases and draw more branch-and-bound trees)

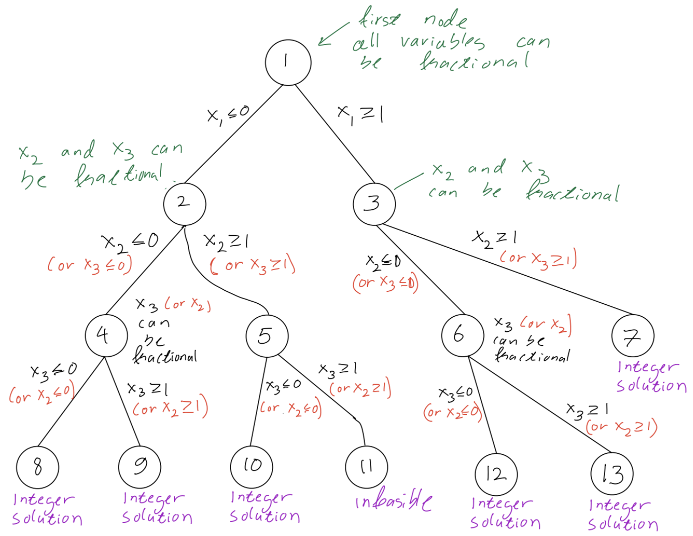

Figure 1: Illustration of possible branch-and-bound tree shape when we start by branching on x_1.

From the figure above, we see that we might have to explore at least 13 nodes. But, could there be a case when we have to explore more than 13 nodes?

Next, we consider the case when we start by branching on either x_2 or x_3 and then branch on x_1. The possible shape of the branch-and-bound tree is illustrated in Figure 2.

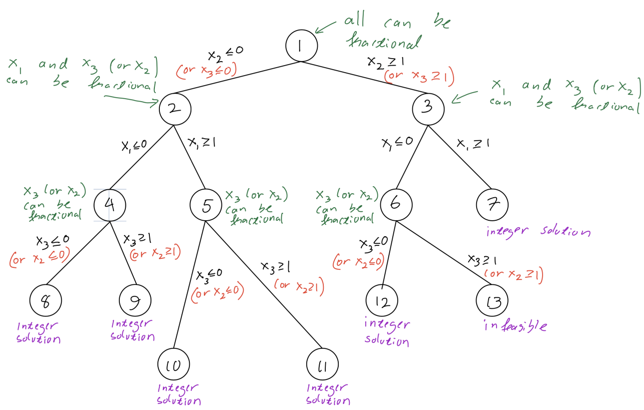

Figure 2: Illustration of possible branch-and-bound tree shape when we start by branching on x_2 or x_3 followed by branching on x_1.

Again, we get a possibility of 13 nodes in the branch-and-bound tree. But, what if we instead of branching on x_1 in node 2 and in node 3 had branched on x_3 or x_2 instead? We can analyze these two cases independently as the branching decision only affects the nodes that follow underneath. Let's see how the sub-trees under node 2 and node 3 change if we instead branched on x_3 or x_2 instead.

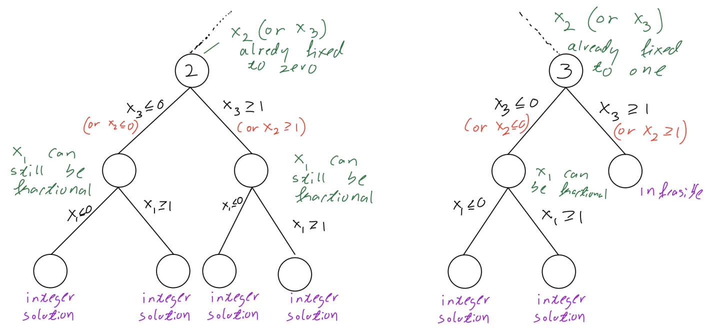

Figure 3: Alternative branching order in the sub-trees following node 2 and node 3 in the branch-and-bound tree in Figure 2.

As illustrated in Figure 3, the total number of nodes remains 13 with the alternative branching orders.

From the analysis of the possible shapes of the brach-and-bound trees, we can conclude that the maximum number of nodes that one might need to explore is 13.

(b) i. The resulting cover-cut is $x_2 + x_3 \le 1$. It is a valid inequality constraint as x_2 and x_3 cannot both be equal to one (only one of them can take the value one). The cover-cut does not exclude any integer feasible solutions. This additional constraint clearly excludes some fractional solutions that satisfy the original constraint. For example, $x_1 = 0, x_2 = \frac{3}{4}, x_3 = \frac{3}{4}$ satisfies the original constraint, but this solution clearly violates the cover cut.

ii. Note that the LP relaxation will contain the constraint $x_1 \le 1$. If we add the cover-cut $x_2 + x_3 \le 1$ to the LP relaxation, then the LP relaxation will always satisfy the constraints

$$x_1 \leq 1,$$
$$x_2 + x_3 \leq 1.$$

The solution to the LP relaxation will, therefore, also satisfy $x_1+x_2+x_3 \leq 2$ (the inequality can be constructed by combining the two constraints above). Thus the first cover-cut ($x_1+x_2 \leq 1$) automatically ensures that the second ($x_1 + x_2 + x_3 \leq 2$) is satisfied in the LP relaxation.
However, satisfying $x_1 + x_2 + x_3 \leq 2$ and $x_1 \leq 1$ does not guarantee that $x_1 + x_2 \leq 1$ is satisfied. For example, consider $x_1 = 0, x_2 = \frac{3}{4}, x_3 = \frac{3}{4}$. Therefore, the cover-cut $x_1 + x_2 \leq 1$ is stronger than $x_1 + x_2 + x_3 \leq 2$.