SF2812 Applied linear optimization, final exam
June 7 2023 08.00–12.00
Brief solutions

1. (a) The transportation problem in the GAMS model can be written as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{2}\sum_{j=1}^{3} c_{i,j}x_{i,j} \\ \text{subject to} & \sum_{j=1}^{3} x_{i,j} \leq a_i, \quad i=1,2, \\ & \sum_{i=1}^{2} x_{i,j} = b_i, \quad j=1,2,3, \\ & \\ & x_{i,j} \geq 0, \quad i=1,2,\ j=1,2,,3, \end{array}$$

where $c_{i,j}$ are the transportation costs, a_i are the capacities of the factories, and b_j are the demands. In the exam, I expect you to also write out the values of $c_{i,j}$, a_i , and b_j.

(b) From the GAMS output we can get the solution

$$x = \begin{pmatrix} 100 & 300 & 0 \\ 225 & 0 & 275 \end{pmatrix},$$

and dual variables

$$s = \begin{pmatrix} 0 & 0 & 40 \\ 0 & 10 & 0 \end{pmatrix},$$

and

$$y = \begin{pmatrix} 0 \\ 0 \\ 250 \\ 170 \\ 140 \end{pmatrix}.$$

Here s are the dual variables associated with the lower bounds of x and y contains the dual variables associated with the constraints from the supply limits and demands.

(c) Since the dual variable associated with the capacity limiting constraint of factory f1 is zero, increasing the capacity would not improve the objective function. (This is a bit unexpected, as all the capacity of factory f1 is used.)

(d) As long as the reduced costs are positive the solution will not change. Thus, we can analyze how much the costs need to decrease for the solution to change. The reduced costs are given by

$$c_N^T - c_B^T B^{-1} N.$$

2. See Lecture note 7, or Chapter 10.2 in Griva, Nash, and Sofer (pages 321 – 324).

3. The basis corresponding to $\tilde{y}$ and $\tilde{s}$ is $\mathcal{B} = \{1, 2\}$.It is straightforward to verify that $B^T y = c_B$ and $s = c - A^T y \geq 0$. Hence, y and s are dual feasible. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

giving us $x = [6 \ -1 \ \ 0 \ \ 0]^T$. Since $x_2 < 0$, the solution is not optimal. We will then start by selecting x_2 as a new nonbasic variable.

Next, we can determine the search direction q (for the y-variables) from

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

resulting in $q_1 = 1, q_2 = -1$.

We can easily determine the search direction η_N (for the nonbasic s-variables), as $\eta_3 = -1, \eta_4 = 1$. The maximal step length $\alpha_{max} = 1$. Updating the variables give us

$$y = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

$$s = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

with the new basis $\mathcal{B} = \{1, 3\}$. The basic x-variables are then given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

with the solution $x_1 = 4, x_3 = 1$. The solution is thus primal feasible and as it is dual feasible it is also optimal. The optimal solution is

$$x = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, y = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, s = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

4. (a) Here, an important observation is that there are no constraints linking the variables x_1, x_2 with the variables x_3, x_4. Therefore, we can consider the problem

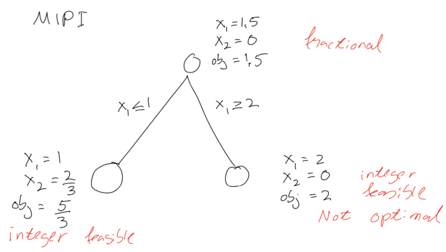
$$
(MIP) \quad \begin{array}{ll} \text{minimize} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + \frac{3}{4}x_2 \geq \frac{3}{2}, \\ & \frac{4}{5}x_3 + x_4 \leq 2, \\ & 0 \leq x_j \leq 3, \quad j = 1, \dots, 4, \\ & x_1, x_3, x_4 \in \mathbb{Z} \quad \text{(integer variables).} \end{array}
$$

as the two separate problems

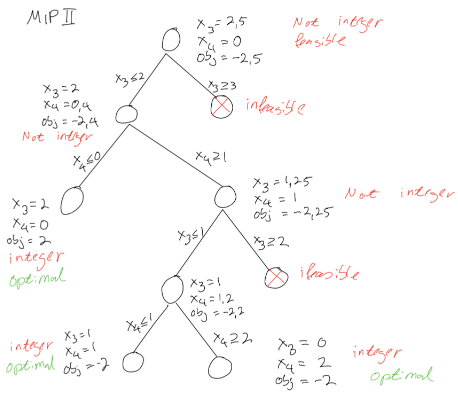
$$
(MIP1) \quad \begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 + \frac{3}{4}x_2 \geq \frac{3}{2}, \\ & 0 \leq x_j \leq 3, \quad j = 1, 2 \\ & x_1 \in \mathbb{Z} \end{array}
\qquad
(MIP2) \quad \begin{array}{ll} \text{minimize} & -x_3 - x_4 \\ \text{subject to} & \frac{4}{5}x_3 + x_4 \leq 2, \\ & 0 \leq x_j \leq 3, \quad j = 3, 4, \\ & x_3, x_4 \in \mathbb{Z} \end{array}
$$

You can either use this to solve it as two separate problems with branch-and-bound resulting in two branch-and-bound trees, or solve it as a single branch-and-bound tree where you use the fact that the problem can be split into two problems to solve subproblem at each step of branch-and-bound. A problem with 4 variables is tricky with pen and paper, but a problem with 2 variables is easy(and two such problems is still easy).

Here I solve it as two separate problems. First, I consider MIP1 and get the following branch and bound tree



And for MIP2 we get the following tree



Note that MIP2 has 3 optimal solutions.

(b) The problem has 3 equally good solutions. The optimal solutions are $x_1 = 1, x_2 = \frac{2}{3}$ with either $x_3 = 2, x_4 = 0$ or $x_3 = 0, x_4 = 2$ or $x_3 = 1, x_4 = 1$.

(c) Rounding down the coefficients on the left-hand side only makes the constraint weaker as the variables are positive. Since all the variables in the constraint are integer-valued, we know that the left-hand side must take integer values for all integer feasible solutions. Therefore, we can round down the right-hand side as it will not exclude any integer feasible solution to the problem. Thus, the Chvatal Gomory cut will not exclude any integer feasible solutions.

(d) We get the Chvatal Gomory cut

$$x_3 + x_4 \leq 2.$$

Now, if we look at the branch-and-bound tree we obtained for MIP2 we can see that this cut is violated by all the fractional solutions we found. The cut would, therefore, clearly eliminates fractional solutions to the problem.

In fact, if we add this Chvatal Gomory cut to MIP2 we directly get an integer solution at the first node of the branch-and-bound tree.

5. (a) For problem P1 the optimal solution is to invest as much as possible in the asset with the highest expected return, i.e., choosing $x_i = 1$ for asset i with the largest r_i. Investing in any other asset will result in a lower expected return.

(b) For problem P2, an important observation is that the lower-level problem

$$\underset{r \in P}{\text{minimize}} \;\; r^T x$$

has the optimal solution that $r = \underline{\mathrm{r}}$ no matter how x is chosen in the first level. For problem P2 the worst possible outcome is always that all returns r_i take their smallest value. Therefore, the optimal solution is to invest everything in asset i with the largest $\underline{\mathrm{r}}_i$ (the asset with the best lower bound on the expected return). Investing any amount in another asset will result in a lower expected return as the lower level problem can always set all $r_i = \underline{\mathrm{r}}_i$.

(c) With the additional constraint all returns r_i cannot be at their lower bound. For example, selecting all $x_i = \frac{1}{100}$ might then be a better solution as only some return r_i can be at their lower bound and some must be higher. However, the optimal solution depends on the returns and their bounds. This situation is more interesting as we are now assuming not all assets will perform as poorly as possible.

(d) By following the hint it should be clear that we want to form the dual of the lower-level problem

$$\underset{r \in N}{\text{minimize}} \;\; r^T x.$$

We start by rewriting the lower-level problem as

$$\begin{array}{ll} \underset{r}{\text{minimize}} & x^T r \\ \text{subject to} & \sum_{i=1}^{100} r_i \;\; - \tilde{r} \geq 0, \\ & r_i - \underline{\mathrm{r}}_i \geq 0, \;\; i = 1, \dots 100, \\ & \bar{r}_i - r_i \geq 0, \;\; i = 1, \dots 100. \end{array}$$

This is the problem that we want to form the dual of. We can start by defining the Lagrangian dual function as

$$\varphi(\lambda,\alpha,\beta) = \underset{r}{\text{minimize}} \ \sum_{i=1}^{100} x_i r_i - \lambda(\sum_{i=1}^{100} r_i - \tilde{r}) - \sum_{i=1}^{100} \left(\alpha_i(r_i - \underline{r}_i) + \beta_i(\bar{r}_i - r_i)\right),$$

and the Lagrangian dual problem is then given by

$$\begin{array}{ll} \underset{\lambda,\alpha,\beta}{\text{maximize}} & \varphi(\lambda,\alpha,\beta) \\ \text{subject to} & \alpha_i \geq 0, \ i = 1,\dots 100, \\ & \beta_i \geq 0, \ i = 1,\dots 100. \end{array}$$

By rearranging the terms in the Lagrangian dual function we get

$$\varphi(\lambda,\alpha,\beta) = \underset{r}{\text{minimize}} \ \lambda\tilde{r} + \sum_{i=1}^{100} (\alpha_i \underline{r}_i - \beta_i \bar{r}_i) + \sum_{i=1}^{100} (x_i + \beta_i - \lambda - \alpha_i)\, r_i,$$

which we write as

$$\varphi(\lambda,\alpha,\beta) = \underset{r}{\text{minimize}} \ \lambda\tilde{r} + \underline{r}^T\alpha - \bar{r}^T\beta + \sum_{i=1}^{100} (x_i + \beta_i - \lambda - \alpha_i)\, r_i.$$

In this case we can write the Lagrangian dual function explicitly as

$$\varphi(\lambda,\alpha,\beta) = \begin{cases} \lambda\tilde{r} + \underline{r}^T\alpha - \bar{r}^T\beta, & \text{if } x_i + \beta_i - \lambda - \alpha_i = 0 \ \forall i = 1,\dots,100 \\ -\infty, & \text{else.} \end{cases}$$

Thus, the dual of the inner problem is

$$\begin{array}{lll} \underset{\lambda,\alpha,\beta}{\text{maximize}} & \lambda\tilde{r} + \underline{r}^T\alpha - \bar{r}^T\beta & \\ \text{subject to} & x_i + \beta_i - \lambda - \alpha_i = 0, & i = 1,\dots 100, \\ & \alpha_i \geq 0, & i = 1,\dots 100, \\ & \beta_i \geq 0, & i = 1,\dots 100. \end{array}$$

As we have strong duality, we know that the dual problem will have the same optimal objective value as the primal and we can choose to solve the dual instead of the primal. If we use the dual form of the lower level then we have maximization in each step and we can combine it into the single problem

$$\begin{array}{lll} \underset{x,\lambda,\alpha,\beta}{\text{maximize}} & \lambda\tilde{r} + \underline{r}^T\alpha - \bar{r}^T\beta & \\ \text{subject to} & \sum_{i=1}^{100} x_i = 1, & \\ & x_i + \beta_i - \lambda - \alpha_i = 0, & i = 1,\dots 100, \\ & x_i \geq 0, & i = 1,\dots 100, \\ & \alpha_i \geq 0, & i = 1,\dots 100, \\ & \beta_i \geq 0, & i = 1,\dots 100. \end{array}$$