SF2812 Applied linear optimization, final exam
Thursday March 14 2024 08.00–13.00

Examiner: Jan Kronqvist, tel. 08 790 71 37.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what has been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

---

1. A person called JK needs to solve a production planning problem, and he has modeled it as an LP problem in GAMS and solved it. However, JK is not very good at using GAMS and is a bit uncertain about the model. JK needs your help to understand the model and to analyze the result.

You can find the GAMS model and GAMS output on the next two pages.

(a) Based on the GAMS model, write down the mathematical formulation of the LP problem. ....(2p)

(b) From the GAMS output, find and write down the optimal primal and dual variable values. ....(2p)

(c) Help JK figure out which resources are limiting the production at the optimal solution, and which resource would be most beneficial to increase the limit of. Carefully motivate your answer, but no calculations are needed ....(2p)

(d) If a small amount of either product 2 or product 3 must be produced, which one would you choose to produce? No calculations are necessarily needed, but carefully motivate your answer. ....(2p)

(e) How much smaller would the objective coefficients of product 2 and product 3 need to be before the optimal solution changes? ....(2p)
Hint: The calculations needed should be quite simple with the information you get from the GAMS output.

GAMS model

```
Sets
P  /product1, product2, product3, product4/
R /raw_material1, raw_material2, time/;

Parameter Limits(R) Limits of different resources
/   raw_material1  15
    raw_material2  10
    time           80/;


Parameter total_cost(P) cost of production minus selling price for each product
/   product1  -8
    product2  -7
    product3  -8
    product4  -20/;

TABLE   Data(P,R) how much reseources are used for each product
                raw_material1   raw_material2   time
    product1    2               4               9
    product2    4               1               7
    product3    5               7               8
    product4    8               4               15;

Variables obj_variable        the objective function variable:
Positive variables  x(P)      amounts produced of each product
                    xe(R)     excess of the resources (slack variable);

Equations
objective                  Objective function
resource_constr(R)         Resource constraints;


objective.. sum(P, total_cost(P)*x(P)) =E= obj_variable;
resource_constr(R).. sum(P, Data(P,R)*x(P)) +xe(R) =E= Limits(R);


Model prod_plan /ALL/;
Solve prod_plan using LP minimizing obj_variable;
```

GAMS output

```
                S O L V E      S U M M A R Y

     MODEL   prod_plan           OBJECTIVE  obj_variable
     TYPE    LP                         DIRECTION  MINIMIZE
     SOLVER  CPLEX               FROM LINE  35

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE              -40.0000

Optimal solution found
Objective:          -40.000000



---- EQU resource_constr  Resource constraints

                     LOWER          LEVEL          UPPER         MARGINAL

raw_material1        15.0000        15.0000        15.0000        -2.0000
raw_material2        10.0000        10.0000        10.0000        -1.0000
time                 80.0000        80.0000        80.0000          .


---- VAR x  amounts produced of each product

                LOWER          LEVEL          UPPER         MARGINAL

product1          .             0.8333        +INF             .
product2          .              .            +INF            2.0000
product3          .              .            +INF            9.0000
product4          .             1.6667        +INF             .

---- VAR xe  excess of the resources (slack variable)

                     LOWER          LEVEL          UPPER         MARGINAL

raw_material1          .              .            +INF           2.0000
raw_material2          .              .            +INF           1.0000
time                   .            47.5000        +INF             .
```

2. Consider a linear program (LP)

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} -4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \end{pmatrix}^T.$$

The dual of problem (LP) is given by

$$(DLP) \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

(a) A friend that you do not really trust claims to have the optimal solution to both (LP) and (DLP). The friend claims that

$$x = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

would be the optimal solution. Show that this cannot be an optimal combination of primal and dual variables .... (2p)

(b) Find the optimal solution to (LP) by using the primal simplex algorithm. Start by setting x_1, x_2, and x_5 as basic variables and perform one iteration with primal simplex. Show that the solution you obtained after just one iteration is optimal. .... (6p)

(To avoid any confusion, by one iteration we mean all the steps and computations needed to move from on basic feasible solution to another)
At some stage of the calculations, you may find it useful to know that

$$\begin{pmatrix} -4 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -0.25 & 0.25 & 0 \\ 0 & 1 & 0 \\ 0.5 & -1.5 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -4 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -0.5 & 0.5 \\ 0 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

(c) Using the knowledge you obtained from solving problem (LP), determine the optimal dual variables y and s. .... (2p)

3. Consider a linear program in standard form

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & A_H x = b_H, & A_H \text{ is "complicating", dimension } m \times n, \\ & A_E x = b_E, & A_E \text{ is "easy"}, \\ & x \geq 0. & \end{array}$$

Assume that $\{x : A_E x = b_E, x \geq 0\}$ is bounded with extreme points v_i, $i = 1, \ldots, k$. Assume further that the problem is solved by Dantzig–Wolfe decomposition.

The master problem becomes

$$\begin{array}{ll} \text{minimize} & c^T V \alpha \\ \text{subject to} & A_H V \alpha = b_H, \\ & e^T \alpha = 1, \\ & \alpha \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{minimize} & \sum_{i=1}^k c^T v_i \alpha_i \\ \text{subject to} & \sum_{i=1}^k A_H v_i \alpha_i = b_H, \\ & \sum_{i=1}^k \alpha_i = 1, \\ & \alpha \geq 0. \end{array}$$

Here e denotes a k-dimensional vector with all components one, and $V = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$.

(a) Derive the subproblem as a linear program. .... (10p)

4. A friend of yours wants to solve the optimization problem

$$(P1)\quad \begin{array}{ll} \text{minimize} & f(x_1, x_2) \\ \text{subject to} & Ax \le b, \\ & -5 \le x_i \le 5, \quad i = 1, 2, \\ & x \in I\!R^n,\ x_i \in \mathbb{Z} \quad \forall i \in \mathcal{I}. \qquad \text{(some variables restricted to integers)} \end{array}$$

(a) Problem (P1) contains the constraint $x_3 + 2x_4 + 2x_5 \le 3$, where the variables x_3, x_4, x_5 are all binary variables. Your teacher JK claims that it would be better to replace that constraint with two new constraints
$x_3 + x_4 + x_5 \le 2$ and $x_4 + x_5 \le 1$.
Show that the new constraints give a stronger continuous relaxation by proving that any solution (fractional or integer) that satisfies the two new constraints also satisfies the single original constraint but not vice versa. ....(4p)

At first glance $(P1)$ looks like a mixed-integer linear programming problem. However, the objective function $f(x_1, x_2)$ is not a linear function. Instead, $f(x_1, x_2)$ is described by the neural network shown in Figure 1.

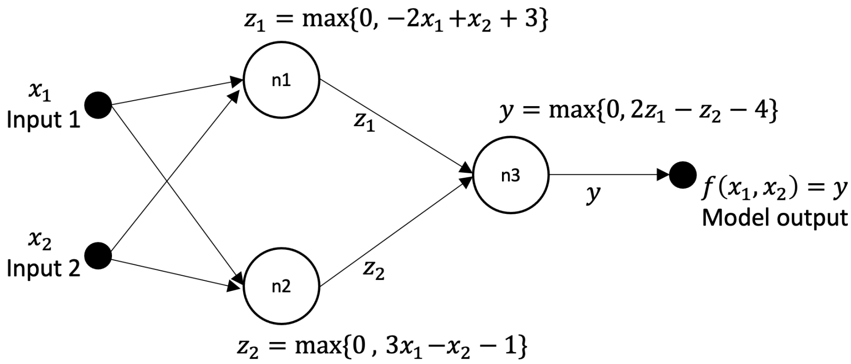

Figure 1: The neural network whose output describes the objective function of $(P1)$.

The neural network has 3 so-called hidden nodes n1, n2, and n3. The output of each hidden node is given by the maximum of 0 and linear function. The hidden node n1 takes variables x_1 and x_2 as inputs, and gives the intermediate variable z_1 as an output given by $z_1 = \max\{0, -2x_1 + x_2 + 3\}$. Similarly, hidden node n2 gives the intermediate variable z_2 as an output. Finally, the node n3 takes the intermediate variables z_1 and z_2 as inputs and returns the output variable y.

By introducing one new binary variable (0 or 1 variable) for each node (alternatively two binary variables for each node depending on how you formulate it), it is possible to exactly model the output of each node by some linear constraints that depends on the inputs to the node and the binary variable. For example, for node n1 we should get some linear constraints that limits the variable z_1 depending on variables x_1, x_2 and a binary variable. We can construct such constraints for all the 3 nodes separately, and get a set of linear constraints that limits the output variable y to the output of the neural network. By introducing some new variables (binary variables for the hidden nodes and the intermediate variables of the neural network) and constraints, we can then formulate problem $(P1)$ as a mixed-integer linear problem.

b) Formulate problem (P1) as a mixed-integer linear problem. .... (6p)
Hint: Remember the output of a node is given by 0 **or** by a linear function (when the linear function is positive). Focus on one node to begin with.

5. In many practical applications we encounter some sort of uncertainty in the optimization problems. Consider the linear program

$$(LP1) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & x \in I\!R^n. \end{array}$$

There are applications where the coefficients of the A matrix and b vector are deterministic and we know exactly the values of these, but there can be uncertainty in how the x variables are implemented in the real-world. For example, there are important optimization problems where the x variables represent setpoints to a machine that doesn't have perfect accuracy. To deal with such inaccuracies, we would prefer to obtain a robust solution where the x-variables can vary slightly and still satisfy all the constraints.

To find a solution that is robust towards small changes in the x variables, we want to select x such that all points within a small neighborhood of x also satisfies the constraints. Such a robust version of the linear program $(LP1)$ can be written as

$$(R-LP) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & c^T x \\ \text{subject to} & \left(\min\limits_{||u||_1 \leq \epsilon} a_i^T(x-u) \right) \geq b_i, \quad i = 1, \ldots m, \end{array}$$

where a_i is row i of matrix A, b_i is the i-th element of vector b, and $\epsilon > 0$ is a given robustness radius. Remember, $||\cdot||_1$ is the 1-norm and $||u||_1 = \sum_{i=1}^n |u_i|$. Note that each of the m constraints in $(R-LP)$ contains a minimization problem. A minimizer x of problem $(R-LP)$ is a robust solution to $(LP1)$ in the sense that any perturbation δ, such that $||\delta||_1 \leq \epsilon$, to x will still be feasible.

Note that the optimization problems in the constraints only consider u as variables (and x is considered constant in these problems). We assume, that the feasible set of $(LP1)$ has a non-empty interior and that ϵ is small enough that there exists a solution to $(R-LP)$.

(a) Consider the constraint problems

$$\begin{array}{ll} \text{minimize} & a_i^T(x-u) \\ \text{subject to} & ||u||_1 \leq \epsilon, \\ & u \in I\!R^n. \end{array}$$

i. Formulate the constraint problems in linear programming form without a 1-norm or absolute values. .... (2p)
Hint: The absolute values can, for example, be represented by introducing some additional variables and constraints.

ii. Determine the dual of the constraint problems. .... (4p)

(b) Use your expertise in linear programming and duality to formulate a linear programming problem that is equivalent to $(R-LP)$. The problem must be a linear program without any optimization problems in the constraints and have a finite number of constraints. We assume that the problem is feasible and that strong duality holds for all the problems. .... (4p)
– If you could not solve a), see comment on next page.

If you could not figure out the dual problems in the previous question, simply assume the dual of the constraint problems are

$$\begin{array}{ll} \text{maximize} & a_i^T x + d^T y_i \\ \text{subject to} & B y_i = f_i, \\ & D y_i \geq h, \end{array}$$

with $y_i \in I\!R^k$ as variables. This is not really the correct dual problems, but it captures the main components and allows you to continue.