GMT Exercise Class 7th February.

Question 1. In the proof of Theorem 4.3.6. the equality $\mu = \mu_1 + \nu$ is stated as obvious. If the set A measurable then the equality follows, given that μ_1 (ν) is the restriction of μ to A (A^c). Is it obvious that A is measurable?

Question 2. In step 1 of the of Riesz Representation Theorem we first show that the series $L(\sum_i \phi\alpha_i), \sum_i |L(\phi\alpha_i)|$ are convergent. Is this needed to do the first inequality on the next line?

Question 3. Is the Radon measure we find in Riesz Rep. Thm. unique? If not, is the measure and the function unique as a pair? If so, what space have we characterised (it looks close to the space of distributions)?

Question 4. Could someone explain the "measure steps" of Theorem 4.1.5.? Are we always increasing the size of the set we are measuring? I don't see how.

Question 5. What is the proof that the Hilbert transform is weak type (1,1). (This is pointed out as an example on page 95 in the book.)

Question 6. How does one prove the last statement of Proposition 4.1.2. where the book claims "the last statement follows from the substance of the proof"?

Question 7. Does the function φ selected in the proof of Theorem 4.1.5. always exists?

Question 8. In Proposition 4.1.2: Can the collection of pairwise disjoint balls $B_j \in \mathcal{B}$ be uncountable many? Can one replace the Lebesgue measure with the Hausdorff measure?

Question 9. Under what assumptions could the hardy-littlewood maximal inequality be extended to Borel measures?

Question 10. the hardy littlewood maximal inequality can be used to study the derivative of a measure w.r.t the lebsegue measure. Would it be possible to extend this inequality to other measures than the lebesgue measure, so that one could determine the dirac and semi continuous part of a measure?

Question 11. When applying Riesz theorem on a general topological space which is locally compact and Hausdorff, how can one ensure that the resulting measure is Radon?

Question 12. In the proof of proposition 4.2.13 we apply the Besicovitch covering theorem on $\tilde{\mathcal{B}}$ where $\tilde{\mathcal{B}}$ consists of the balls in $\mathcal{B}$ that are centered in $A \cap C$ and that are entirely contained in U , where U is a bounded open set containing A. However, $\mathcal{B}$ is a family of balls such that every point of A is the center of arbitrarily small balls in $\mathcal{B}$, and so every point of $A \cap C$ should be the center of arbitrarily small balls of $\tilde{\mathcal{B}}$ But isn't this a problem? Since in order to apply Besicovitch (proposition 4.2.1), $\tilde{\mathcal{B}}$ must have the property that the interiror of no ball contains the center of any other. But won't many balls in $\tilde{\mathcal{B}}$ have the same point in $A \cap C$ as its center? Can we define $\tilde{\mathcal{B}}$ in some other way to work around this issue, or am I completely misunderstanding this?

Question 13. Could we fix the proof of Vitali's covering theorem from the book (regarding the closure being covered by the open cover) using inner regularity of the Lebesgue measure?

Also, what happens to overlapping balls as we glue together cubes to move from bounded sets to unbounded sets?

Question 14. In John's proof of Vitali's covering theorem, it appears

we apply Vitali's covering lemma to ensure that the sequence of balls we construct satisfies the following property:

$$\forall m \in \mathbb{N} : A \setminus (\cup_{j=1}^{m} B_j) \subset \cup_{k=m+1}^{\infty} B_k \tag{1}$$

I understand how this works for a fixed m,

but as we let m increase the set of balls we get from the covering lemma could change at every step.

How do we ensure this does not happen?

Question 15. About spherical measure: How to show the spherical measure of $E = [0,1] \cap \mathbb{Q}$ is 0 using the definition ?

Though we have a nice inequality that $\mathcal{H}^m \leq \mathcal{S}^m \leq 2^m \mathcal{H}^m$ with $\mathcal{H}^m(E) = 0$, since we could simply take singletons whose diameters are zero. However, I am not sure if we allow balls have radius zero in the spherical measure. If not, it is not obvious for me to show $\mathcal{H}^m(E) = 0$.

Question 16. Why we need the definition of weak type and weak type (p,p)? Recall that Markov's inequality says for any $f \in L^p(\mu)$

$$\mu(\{x \in X \;:\; |f(x)| \geq t\}) \leq \frac{1}{t^p} \int |f|^p \, d\mu \tag{2}$$

therefore such f is of weak type p. It looks like weak type generalises this kind of property, which gives a nice bound for the potentially nasty set. But why it is called weak?

Question 17. Is the proof of Prop 4.1.4 is correct? Or is this is the following a counterexample (?):

$$f(x) = 1 \implies Mf(x) = 1$$

$$\lambda = 1/2 \implies S_\lambda = \mathbb{R}^n$$

Then there exists no compact K such that

$$\infty = \mathcal{L}^n(S_\lambda) < 2\mathcal{L}^n(K) < \infty$$

If this is a counterexample can we the argument so that it works?

Question 18. Covering/topological dimension of $\mathbb{R}^n$

At page 91 the book says:

"We know, thanks to Lebesgue, that any open covering of $S \subset \mathbb{R}^n$ has a refinement with valence at most $N+1$.

Is there a quick and easy proof?

Question 19. Lebesgue measure is Radon: Prove that the Lebesgue measure in $\mathbb{R}^n$ is a Radon measure (when defined as the product measure of one-dimensional Lebesgue measures).

Question 20. Remark 4.3.1:
This remark is a sketch of a proof as to why the upper and lower derivatives of Radon measures are Borel measurable. Something seems to be wrong about it. Let μ be some Radon measure on $\mathbb{R}^n$. They start off the proof by saying that for any $R > 0$ the function $f_{(\mu,R)} : \mathbb{R}^n \to \mathbb{R}$

$$f_{(\mu,R)}(x) = \mu(B(x,R)) = \int_{B(x,R)} d\mu$$

is continuous. $B(x,R)$ is an open (or closed?) ball of radius R around x.

Let us construct a counterexample. If we let δ be the dirac measure (a set has measure 1 iff it contains 0 and has 0 measure otherwise). This is a Radon measure. Note that then

$$f_{(\mu,R)} = \mathbf{1}_{B(0,R)}$$

which is not continuous regardless if the balls are closed or open. It is continuous in a lot of points though.

Can you debunk the counterexample, fix the proof or find an alternate proof?

Question 21. Prove density of continuous functions in L^1
Can you prove or give some intuition as to why the continuous functions are dense in L^1? In what other classes of functions are they dense?

Question 22. When proving Vitali covering lemma we have assumed (w.l.o.g.) that the set which we want to cover with open balls has finite Lebesgue measure. How can we prove the covering lemma for sets of infinite measure? Note that it is not clear that we can "glue together" covers of sets of finite measure to get the cover of the set with infinite measure, since then we can not ensure that balls do not intersect.

Question 23. In Vitali covering lemma we can prove that $\bigcup_j 3B_j$ are covering our set U_0. Show that the constant 3 is tight.

Question 24. Riesz representation theorem gives us a bijection between the set of bounded linear functionals on the set of compactly supported continous functions on $\mathbb{R}^n$, and the set of Radon measures on $\mathbb{R}^n$. Therefore, the assumptions for Riesz representation theorem fail in case we consider linear operator

$$\Lambda_\mu \colon C_c(\mathbb{R}^n) \to [0,\infty), \quad \Lambda_\mu(f) = \int_{\mathbb{R}^n} f d\mu,$$

when μ is not a Radon measure. Can you find such measure[1] μ, and which conditions fails?

[1]In particular, Borel measure.