GMT Exercise Class 6th Mars.

Question 1. Is the union of rectifiable sets necessarily rectifiable?

Question 2. Can one provide with an example on how to find approximate tangent spaces to some concrete curve.

Question 3. In definition 6.2.2, what do they mean with:

The $(N-1)-$dimensional rectangular solid $\mathcal{F} \subset \mathbb{R}^N$ will be oriented by the $(N-1)-$vector

$$\hat{\mathbf{e}}_i = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{i-1} \wedge \mathbf{e}_{i+1} \wedge \cdots \wedge \mathbf{e}_N.$$

What is the geometric intuition?

Question 4. From the area formula we know that "the area" of a parametrizable manifold coincides with its Hausdorff measure, where in formula (5.1) $f : A \subset \mathbb{R}^M \to \mathbb{R}^N$ stands in this case for the parametrization. Would it be possible to extend this result to orientable manifolds making use of the differentiable structure, meaning that we could try to work with the charts and then "patch things together"?

Remark Integration of differentiable forms is already something well defined on oriented manifolds, but if we try to answer the question above using this, then what we are really asking is which differentiable form should be chosen so that a similar result to the area formula holds.

Question 5. Define $\bar{B}^3 = \{x \in \mathbb{R}^3, |x| \leq 1\}$ with the usual euclidean norm, and so notice that $\partial \bar{B}^3 = \mathbb{S}^2$. Using spherical coordinates $(\varphi, \theta) \in \bar{D}$ with $D = (0, \pi) \times (0, 2\pi)$, compute the integral

$$\int_{\bar{B}^3} d\omega$$

using Stokes Theorem, where $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$. Notice that the parametrization of $\mathbb{S}^2$ in these coordinates can be written explicitly as $F : \bar{D} \to \mathbb{R}^3$,

$$F(\varphi, \theta) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi).$$

Question 6. Give an example of a C^1 differential form so that $d^2\omega \neq 0$.

Question 7. If one for some reason is unhappy about the definition of exterior product given on page 160, then is here an alternative definition based on the tensor product. Given m co-vectors $a_i \in \Lambda^1\mathbb{R}^n = \{a : \mathbb{R}^n \to \mathbb{R} \text{ linear}\}$, define the exterior product of the a_i as

$$a_1 \wedge a_2 \wedge \cdots \wedge a_m := \frac{1}{m!} \sum_{\sigma \in \mathcal{P}_m} \operatorname{sgn}(\sigma) a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(m)}$$

where $\mathcal{P}_m$ is the set of all permutations of the set $\{1, 2, \ldots, m\}$ to itself. Since $[a_1 \otimes a_2 \otimes \cdots \otimes a_m](u_1, \ldots, u_m) = a_1(u_1)a_2(u_2)\ldots a_m(u_m)$ is already multilinear, we see that the sum of permutations in the above is done to establish the alternating property. Thus $a_1 \wedge a_2 \wedge \cdots \wedge a_m \in \Lambda^m \mathbb{R}^n$.

Moreover, from this definition and the definition of the determinant, it is clear to see the validity of equation (6.2) on page 161.

Question 8. Could someone elaborate on why the definition (6.4) on page 162 is natural? It is relatively clear that $\phi \circ F$ will be a differential m-form, and that for each point $\frac{\partial F}{\partial t_i}$ will be a vector. So pointwise I guess the formula makes sense if we extend the definition of m-form to act on functions, but I still have no idea what it means.

Question 9. Let $\omega_1, ..., \omega_K$ be $M-$forms in $\mathbb{R}^N$ then we may define the non-linear integration with respect to $F(\omega_1, \omega_2, .., \omega_K)$ according to

$$\int_S F(\omega_1, \omega_2, .., \omega_K) =$$

$$= \int_U F\left(\left\langle \omega_1 \circ F(t), \frac{\partial f}{\partial t_1} \wedge \cdots \wedge \frac{\partial f}{\partial t_M} \right\rangle, ..., \left\langle \omega_K \circ F(t), \frac{\partial f}{\partial t_1} \wedge \cdots \wedge \frac{\partial f}{\partial t_M} \right\rangle\right) d\mathcal{L}^M(t),$$

where S is parametrized by the C^1 function $f : U \mapsto \mathbb{R}^N$.

Given this, what is the geometric meaning of

$$\int_S \sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dx \wedge dz)^2},$$

where S is a two dimensional C^1-surface in $\mathbb{R}^3$.

Question 10. We have discussed two different ways of integrate on surfaces S given by a C^1 mapping $F : U \subset \mathbb{R}^M \mapsto \mathbb{R}^N$, $1 \leq M \leq N$, lets for simplicity assume that F is one to one. First we have the *Area formula (Theorem 5.1.1):*

$$\int_U J_M f(x) d\mathcal{L}^M(x) = \int_{\mathbb{R}^N} \mathrm{card}(A \cap f^{-1}(y)) d\mathcal{H}^M(y) = \mathcal{H}^M(S). \tag{1}$$

and integrating an $M-$form ϕ over the surface *(formula (6.4))*

$$\int_S \phi = \int_U \left\langle \phi \circ F(t), \frac{\partial F}{\partial t_1} \wedge \cdots \wedge \frac{\partial F}{\partial t_M} \right\rangle d\mathcal{L}^M(t). \tag{2}$$

Are the two definitions the same? Can we find a form ϕ so that (1) and (2) are the same for all surfaces S?

Question 11. This question is very related to the boundary of currents, and may be viewed as a prelude to the section on currents. Given a (countably) M-rectifiable set E. We want to define ∂E. The literature seems to give an ad-hoc definition of boundary. Supposedly, ∂E is the $M-1$-rectifiable set such that

$$\int_{\partial E} \omega = \int_E d\omega$$

for all C^1 differential $M-1$-forms ω. What does it mean for ω to be differentiable on a rectifiable set? Also, are there any further restrictions on E in order for this definition to be possible?

Question 12. The idea of the bounded variation for a continuous function of a single variable is that the distance along the direction of y is finite. The formal definition is as for $f : a, b \to \mathbb{R}$

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)| < \infty$$

By introducing the divergence of test functions, how does the definition of (local) bounded variation for multivariate function (5.30 in page 151) generalise the idea as above? Moreover, can we have similar results as theorem 5.5.6 and theorem 5.5.7 (page 155-156) for u of bounded quadratic variation rather than bounded variation? (I was thinking of Brownian path which is of bounded quadratic variation but unbounded variation. But it seems impossible since there is no notion of weak derivatives for functions of unbounded variations.)

Question 13. (Different ways to convergence). Is there a sequence of rectifiable sets $S_j \subset \mathbb{R}^3$ such that

$$S_j \to \overline{Q_1(0)} = \{x \in \mathbb{R}^3;\ |x_1| \le 1 \text{ for } i = 1, 2, 3\}$$

but

$$\int_{S_j} f(x) d\mathcal{H}^2(x) \to \int_{\{x_3=0\} \cap \overline{Q_1(0)}} f(x) d\mathcal{H}^2(x)$$

for any function $f \in C_c(\mathbb{R}^3)$.

Is this important, and why?

Question 14. In the proof of Lemma 5.5.2 where we prove the Poincare inequality:

$$\int_U |f - f_u| d\mathcal{L}^N \le c \int_U |Df| d\mathcal{L}^N$$

we assume that U is open , bounded and convex. If we take away the assumption that U is bounded (or convex) is the Poincare inequality still true?

Question 15. In Theorem 5.5.7 we prove that

$$\int_{\mathbb{R}^n} |Du| d\mathcal{L}^N \le c(U) \int_U (|Du| + |u|) \mathcal{L}^N$$

for BV functions under the assumptoion that U is open, bounded and convex with and $\text{supp}(u) \subset \overline{U}$. Is it still true if U is allowed to be closed? What about if U is not convex?

Question 16. Is the following version of the Poincare inequality true for functions in one variable:

$$\int_I |f(x) - f(a)| dx \le C(I) \int_I |f'(x)| dx$$

where I is an intervall in $\mathbb{R}$ and $a \in I$?

How would you prove it, how does your proof relate to the proof of the general Poincare inequality for BV functions in $\mathbb{R}^N$?

Question 17. If we define the semi-norm on BV:

$$[u]_{BV} = \sup_{|\vec{g}|\leq 1} \int_{\mathbb{R}^n} u(x)\mathrm{div}(\vec{g})(x)d\mathcal{L}^N(x),$$

where the supremum is taken over functions $g \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$, so that the norm of BV becomes $\|u\|_{BV} = \|u\|_{L^1(\mathbb{R}^N)} + [u]_{BV}$.

Assume that $u^j \in BV$ is a sequence of functions such that $u^j \to u \in BV$ in $L^1(\mathbb{R}^N)$. Assume furthermore that $[u^j]_{BV} \leq 1$. Does it follow that $[u]_{BV} \leq 1$?

Reformulation: *Is the semi-norm lower semicontinuous with respect to convegence in L^1?*

Question 18. Is there a measurabel M-rectifiable set S, of positive measure, such that S does not have a classical tangent space at any point?

Question 19. Remember that we define <u>the</u> approximate tangent space to S at $x \in \mathbb{R}^N$ (Definition 5.4.4 on p. 149) to be the M-dimensional <u>linear space</u> W if, for any $f \in C_c(\mathbb{R}^N)$,

$$\lim_{\lambda\to 0^+} \int_{\lambda^{-1}(S-x)} f(y)d\mathcal{N}^M(y) = \int_W f(y)d\mathcal{H}^M(y). \tag{3}$$

a) If we say that C is an approximate tangent <u>cone</u> if it satisfies the same definition as W - except the assumption that it is a linear space. Find an example of a rectifiable set that has an approximate tangent cone C at a point x but no approximate tangent space at x.

b) If C is an M dimensional approximate tangent cone of S at the origin. Show that C is a cone, that is $C_t = C$ for $t > 0$ where

$$C_t = \{x;\ tx \in C\}.$$

c) If we say that W is <u>an</u> approximate tangent space of S at x if (3) is satisfied for some subsequence $\lambda_k \to 0^+$. Con you find a rectifiable set S that has two different approximate tangent spaces at a point x? If it has two different tangent spaces at a point x will it automatically have infinitely many approximate tangent spaces at that point?

Question 20. Define <u>an</u> approximate tangent cone as in **c)** of the previous question. Is there a $1-$rectifiable set in $\mathbb{R}^2$ that has $y = kx$, for any $|k| \leq 1$, as an approximate tangent space?

Question 21.(Structure of BV functions) It is easy to see that the distributional gradient of a BV function is a measure (to be exact a vector $(\mu_1, ..., \mu_N)$ of measures). But is every (vector of) measure(s) the derivative of a $BV-$function?

Question 22. In order to use the Riesz representation theorem to identify the distributional derivative of $u \in BV$ as a vector measure $\nu(x)\mu(x)$ ($\nu(x) \in \mathbb{R}^N$ and μ a measure) one need to show that the functional

$$\vec{g} \mapsto \int_{\mathbb{R}^N} u(x)\mathrm{div}(\vec{g})(x)d\mathcal{L}^N(x)$$

defined on $C_c^1(\mathbb{R}^N;\mathbb{R}^N)$ actually defines a bounded and linear functional on $C_c(\mathbb{R};\mathbb{R}^N)$. How does one show this, is the extension unique?

Question 23. Assume that a fluid flows in $\mathbb{R}^3$ by constant velocity given by a C^1 vector field on $\mathbb{R}^3$, that is a C^1 function $v : \mathbb{R}^3 \mapsto \mathbb{R}^3$.

We want to define a function, $L(S)$, on the C^1 (two dimensional) surfaces in $\mathbb{R}^3$ such that if S is a surface parametrized by $f : U_f \mapsto \mathbb{R}^3$ then $L(S)$ should give us the volume/(time unit) of fluid that passes through the surface. How would you use differential forms and intefration of forms to define L?

Question 24. (Philosophical question) As mathematicians we take pride, as we should, in that we deal with truth. But what does this mean? In what sense is is the previous question givning you "the truth"? At times during the lectures I change the definitions, and last time I changed a Definition into a lemma, in what sense can we claim that both the theory in the book and the one during the lectures give the <u>same</u> truth?[1]

[1] Descartes famously claimed that he only thought about philosophy for 15 minutes every year (yet he was his ages greatest philosopher). He claimed that the philosophical questions where of lesser importance than scientific questions. But I do think that we all should spend 15 minutes from time to time away from our scientific pursuits and think of the philosophical foundations of our subject. This is your 15 minutes for this year....

GMT questions 4

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Wedge product

Let $a_1 \wedge \ldots \wedge a_m, b_1 \wedge \ldots \wedge b_m, \in \Lambda_m(\mathbb{R}^N)$ be elements of the abstract vector space from definition 1.4.1. Also let and $A, B \in \mathbb{R}^{N \times m}$ be the corresponding matrices such that $A_{ij} = e_i \circ a_j$. For any positive integer n let $[n] = \{1, \ldots, n\}$, we may now define

$$T := \{\lambda : [m] \to [N] \quad \textit{strictly increasing function } \}$$

Then we can define two things

$$e_\lambda := e_{\lambda(1)} \wedge \ldots \wedge e_{\lambda(m)} \in \Lambda_m(\mathbb{R}^N) \qquad (A_\lambda)_{ij} := A_{\lambda(i)j} \in \mathbb{R}^{m \times m}$$

Prove that

$$a_1 \wedge \ldots \wedge a_m = \sum_{\lambda \in T} det(A_\lambda) e_\lambda$$

A direct consequence is that the standard inner product, corresponding to the basis e_λ of $\Lambda_m(\mathbb{R}^N)$, can be computed for simple m-vectors by

$$\langle a_1 \wedge \ldots \wedge a_m, b_1 \wedge \ldots \wedge b_m \rangle := \sum_{\lambda \in T} det(A_\lambda) det(B_\lambda) = det(A^T B)$$

Here we used the Cauchy-Binet formula to simplify the expression into $det(A^T B)$.

Use this to bijectively identify $a_1 \wedge \ldots \wedge a_m$ with an alternating multilinear form

$$a_1 \wedge \ldots \wedge a_m : (\mathbb{R}^N)^m \to \mathbb{R} \qquad a_1 \wedge \ldots \wedge a_m(x_1, \ldots, x_n) = det(A^T X)$$

Find the the alternating multilinear forms on $(\mathbb{R}^N)$ corresponding to e_λ, motivate why they are a basis for the set of alternating multilinear forms. **Prove** (or disprove) that

$$a_1 \wedge \ldots \wedge a_m = b_1 \wedge \ldots \wedge b_m \iff \exists K \in SL(m) \quad A = BK$$

0.1 Approximate tangent space

Prove that "of course" the approximate tangent space coincides with the usual one for some C^1 manifold S in $\mathbb{R}^n$ parametrised by ϕ with $\phi(q) = x$ by defining

$$\phi_\lambda(t) := \frac{\phi(t + q) - x}{\lambda}$$

as a parameterisation of the C^1 manifold $\lambda^{-1}(S - x)$. Use the area formula and study $\lim_{\lambda \to 0^+} \phi_\lambda(\lambda t)$