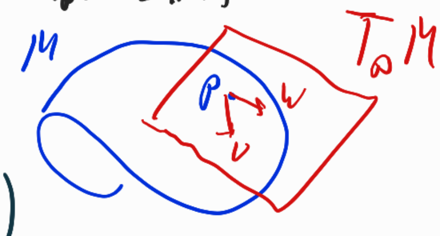
Recap on Riemannian geometry

__Def__ A Riemannian metric g on a mfd M assigns to each $p \in M$ an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$, which depends C^∞ on the point p.



$g_p(v,w) \in \mathbb{R}$

($\forall$ charts (U,φ), $g_{ij} = g(\partial_i, \partial_j) \in C^\infty$)

$g \in \mathcal{T}^0_2(M)$ (g is a $\binom{0}{2}$-tensor field on M)

We call (M,g) a Riemannian manifold.

Sometimes we just write M instead of (M,g) (if g is fixed)
In that case one uses the notation $\langle X,Y\rangle := g(X,Y)$ ($\in C^\infty(M)$) for C^∞ vector fields X,Y on M, i.e. for $X,Y \in \mathfrak{X}(M)$

A Riem metric on M determines the Levi-Civita connection (or covariant derivative)

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$
$$(X,Y) \longmapsto \nabla_X Y$$

Locally if $X|_U = X^i \partial_i$, $Y|_U = Y^j \partial_j$ for a chart (U,φ), we have

$$\nabla_X Y|_U = (X^i \partial_i Y^k + \Gamma_{ij}^{\,k} X^i Y^j)\partial_k,$$

where $\Gamma_{ij}^{\,k} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ (Christoffel symbols)

inverse of g_{ij}

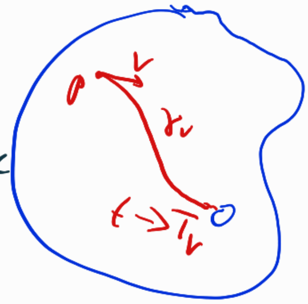
__Ex__ On $(\mathbb{R}^n, g_{eucl})$, $g_{ij} = \delta_{ij}$ in the canonical chart $\Rightarrow \Gamma_{ij}^{\,k} = 0$

Geodesics & Exponential map

<u>Def</u> A curve $\gamma: I \to M$ is a geodesic $\iff \nabla_{\gamma'}\gamma' \equiv 0$

If $\varphi \circ \gamma = (\gamma^1, \dots, \gamma^n)$ in a chart (U, φ), this is equivalent to

$$(\gamma^k)'' + \Gamma_{ij}^k (\gamma^i)' (\gamma^j)' = 0$$



$\Rightarrow$ $\forall p \in M, \forall v \in T_pM$, there exists a unique maximal geodesic

$$\gamma_v : [0, T_v) \to M \quad \text{s.t. } \gamma_v(0) = p,\ \gamma_v'(0) = v$$

<u>Ex</u> ·) On $(\mathbb{R}^n, g_{eucl})$: $\forall p \in \mathbb{R}^n, \forall v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, $\gamma_v(t) = p + t \cdot v$

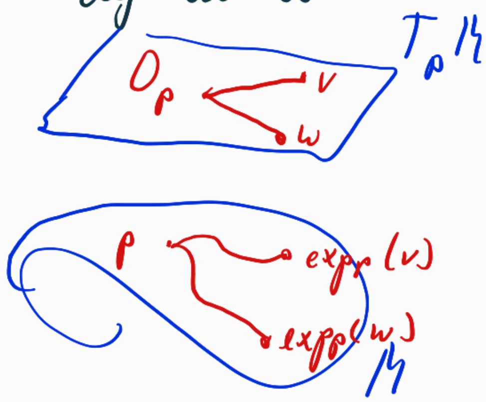
·) If $M \subset \mathbb{R}^m$ is a Riem. submfd ($g = g_{eucl}|_M$), then

$\gamma: I \to M \subset \mathbb{R}^m$ is a geod in $M \iff \gamma''(t) \perp T_{\gamma(t)}M \ \forall t$

<u>Def</u> For $p \in M$, the exponential map $\exp_p$ is defined as

$$\exp_p : T_pM \supset V \to M,$$
$$v \mapsto \gamma_v(1),$$

where $V = \{ v \in T_pM \mid T_v > 1 \}$.



<u>Remark</u> ·) $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$

·) $\exp_p : V \to M$ is a local diffeomorphism around $0 \in T_pM$

$\rightsquigarrow$ $\exists$ nbhd U around p s.t. $(\exp_p)^{-1} : U \to V \subset T_pM \cong \mathbb{R}^n$ is a chart

Curvature

Def The Riemannian curvature tensor is a map
$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$
def as
$$R_{X,Y} Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) + \nabla_{\nabla_Y X} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
Remarkably, R is C^∞-trilinear, hence $R \in \mathcal{T}^1_3(M)$

Often, we regard R as a $\binom{0}{4}$-tensor field by setting
$$R(X,Y,Z,W) := \langle R_{X,Y} Z, W \rangle$$

In local coordinates, $R_{\partial_i, \partial_j} \partial_k = R_{ijk}^{\ell} \partial_\ell$, with
$$R_{ijk}^{\ell} = \partial_i \Gamma_{jk}^{\ell} - \partial_j \Gamma_{ik}^{\ell} + \Gamma_{jk}^{m} \Gamma_{im}^{\ell} - \Gamma_{ik}^{m} \Gamma_{jm}^{\ell}$$

Ex: On $(\mathbb{R}^n, g_{eucl})$ we have $\Gamma_{ij}^k \equiv 0$ in the canonical chart
$$\Rightarrow R_{ijk}^{\ell} \equiv 0 \Rightarrow R \equiv 0$$

Rem: In an exponential chart,
$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{ikj\ell} x^k x^\ell + O(|x|^3)$$

Interpretation: R measures the deviation of the metric from being the flat metric

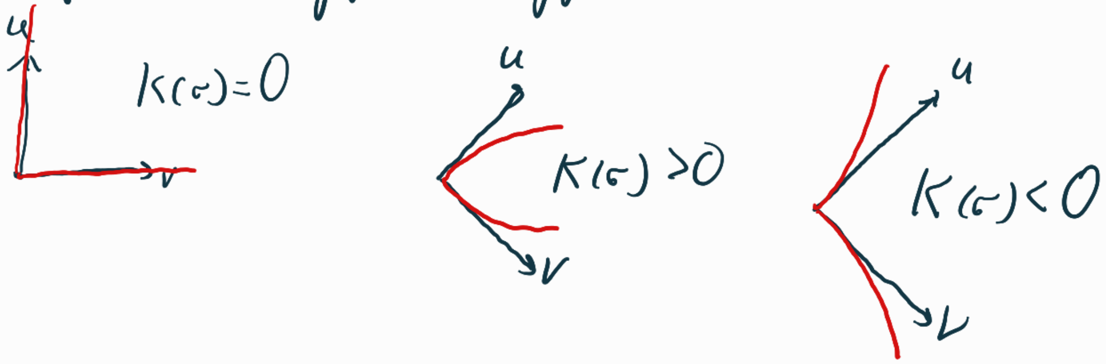
Def For a 2-dim plane $\sigma = \operatorname{span}\{u,v\} \subset T_pM$, the sectional curvature $K(\sigma)$ is

$$K(\sigma) := \frac{R(u,v,v,u)}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}$$

Note: This is indep. of the choice of basis of σ

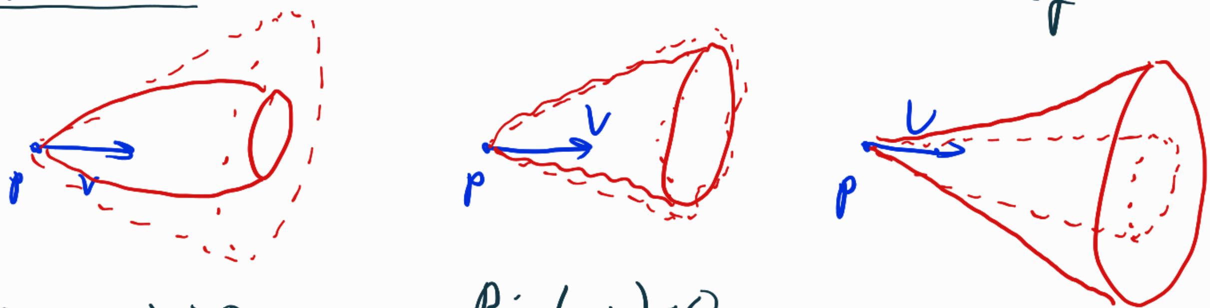
Interpretation: If $\{u,v\}$ is an orthonormal basis of $\sigma \subset T_pM$,

$$d(\gamma_u(t), \gamma_v(t)) = d(\exp_p(tu), \exp_p(tv)) = t^2\|u-v\|^2 - \tfrac{1}{3}K(\sigma)t^4 + O(t^5)$$



Def The Ricci tensor is $\operatorname{Ric}(v,w) = \operatorname{tr}(u \mapsto R_{u,v}w)$ for $v,w \in T_pM$ $\rightsquigarrow \operatorname{Ric} \in \mathcal{T}^0_2(M)$

Interpretation: Volume distortion in the direction of v.



$\operatorname{Ric}(v,v) > 0$ $\quad$ $\operatorname{Ric}(v,v) = 0$ $\quad$ $\operatorname{Ric}(v,v) < 0$

Def The scalar curvature is the function $\operatorname{scal} = \operatorname{tr}\operatorname{Ric} \in C^\infty(M)$

Interpretation: $$\operatorname{vol}(\underbrace{B_r(p)}_{\subset M}) = \left(1 - \frac{\operatorname{scal}}{6(n+2)} r^2 + O(r^4)\right)\operatorname{vol}(\underbrace{B_r(0)}_{\subset \mathbb{R}^n})$$