1.3 Complete Riemannian manifolds and the Hopf-Rinow theorem

<u>Def 1.3.1</u> A Riem. mfd (M,g) is called geodesically complete if $\forall p \in M$, $\exp_p$ is defined on all of T_pM (the maximal geodesics are all defined on $\mathbb{R}$)

<u>Thm 1.3.2</u> (Hopf-Rinow) Let (M,g) be a connected Riem mfd and $p \in M$. The following are equivalent:

(i) $\exp_p$ is def. on T_pM

(ii) Closed and bounded subsets of M are cpt (Heine-Borel property)

(iii) M is complete as a metric space

(iv) M is geodesically complete

If (i)-(iv) hold, we also have

(v) For every $q \in M$, there exists a geodesic $\gamma \in \Omega_{p,q}$ s.t. $L(\gamma) = d(p,q)$

<u>Remark 1.3.3</u> ·) (i), (iv), (v) make sense for semi-Riem. mfds, but (ii)-(iii) do not.
The implication (iv) $\Rightarrow$ (v) does not hold in the semi-Riem case.
De-Sitter space is a counterexample

·) (v) does not imply (i)-(iv) (Convex sets in $\mathbb{R}^n$)

·) Geodesic and metric completeness are equivalent for Riem mfds according to this theorem. From now on, we call these mfds just complete

<u>Proof of Thm 1.3.2</u>

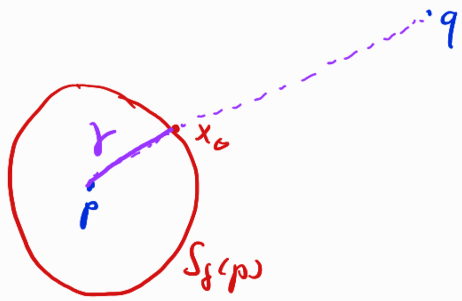
<u>(i) $\Rightarrow$ (v)</u> Let $d(p,q) = r$ and $B_\delta(p)$ a geodesic ball around p.

Let $S_\delta(p) := \partial B_\delta(p)$ $(= \exp_p(\partial B_\delta(0))$, hence cpt$)$

$x \mapsto d(q,x)$ has a minimum on $S_\delta(p)$ in a pt x_0

$\Rightarrow x_0 = \exp_p(\delta v)$ for $v \in T_pM$ with $\|v\| = 1$

Let $\gamma(t) = \exp_p(tv)$. We will show that $\gamma(r) = q$



Let $A = \{ s \in [0,r] \mid d(\gamma(s), q) = r - s \}$.

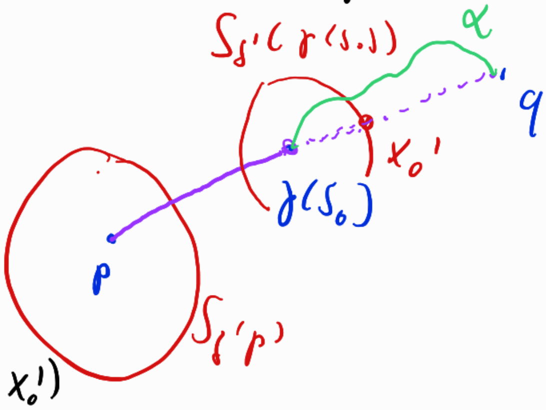
If $A = [0,r]$, then $\gamma(r) = q$, $\gamma|_{[0,r]} \in \Omega_{p,q}$ and $L(\gamma|_{[0,r]}) \overset{\|\dot\gamma\| = 1}{=} r = d(p,q)$

A is closed, so it remains to show that A is open

Let $s_0 \in A$ and $B_{\delta'}(\gamma(s_0))$ a geodesic ball around $\gamma(s_0)$

Let $S_{\delta'}(\gamma(s_0)) = \partial B_{\delta'}(\gamma(s_0))$

$x \mapsto d(q,x)$ has a minimum on $S_{\delta'}(\gamma(s_0))$ in a pt x_0'



Step 1 $d(\gamma(s_0), q) = \delta' + d(x_0', q)$

"$\leq$" Triangle inequality, as $\delta' = d(\gamma(s_0), x_0')$

"$\geq$" $\alpha \in \Omega_{\gamma(s_0),q}$, $\alpha \in [a,b] \to M$ has t_0 s.t. $\alpha(t_0) \in S_{\delta'}(\gamma(s_0))$

$$\begin{aligned} L(\alpha) &= L(\alpha|_{[a,t_0]}) + L(\alpha|_{[t_0,b]}) \\ &\geq \delta' + d(\alpha(t_0), q) \\ &\geq \delta' + d(x_0', q) \end{aligned}$$

α arbitrary $\Rightarrow$ $d(\gamma(s_0), q) \geq \delta' + d(x_0', q)$ ✓

Step 2 $d(p, x_0') = d(p, \gamma(s_0)) + d(\gamma(s_0), x_0') = s_0 + \delta'$

"$\leq$" Δ-ineq

"$\geq$" $d(p, x_0') \geq d(p,q) - d(q, x_0') = r - \overbrace{d(\gamma(s_0), q)}^{r - s_0\ (s_0 \in A)} + \delta'$

$= s_0 + \delta'$ ✓

Step 3 $x_0' = \gamma(s_0 + \delta')$

Let σ be the minimizing geodesic from $\gamma(s_0)$ to x_0'. $L(\sigma) = \delta'$

$\Rightarrow$ $\gamma|_{[0,s_0]} \cup \sigma \in \Omega_{p,x_0'}$ has length $s_0 + \delta' = d(p, x_0')$

$\Rightarrow$ this curve is minimizing, hence a geodesic ✓

From Step 1, we know already

$$d(\gamma(s_0+\delta'), q) = d(x_0', q) = d(\gamma(s_0), q) - \delta' = r - s_0 - \delta'$$

$\Rightarrow$ $s_0 + \delta' \in A$, so A is open

(I) $\Rightarrow$ (II) Let $A \subset M$ be closed and bounded, i.e $A \subset B_R(p)$ for some $R>0$

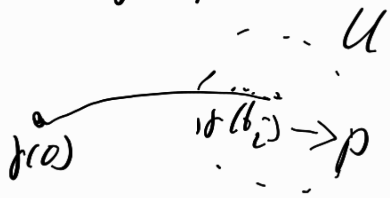
$\overset{(V)}{\Longrightarrow} \forall q \in B_R(p)$ we have a geodesic $\gamma \in \Omega_{p,q}$ s.t $L(\gamma) \leq R$

$\Rightarrow \quad A \subset B_R(q) \subset \underbrace{\exp_p(\overline{B_R(0)})}_{\text{cpt. as } \overline{B_R(0)} \text{ kp, } \exp_p \text{ cont}} \Rightarrow A$ is cpt as closed subset of a cpt set

(II) $\Rightarrow$ (III) $\{p_n\}$ Cauchy-sequence in $M \Rightarrow \{p_1, p_2, \dots\}$ bounded, hence its closure is compact $\Rightarrow \{p_n\}$ contains a conv. subsequence

$\Rightarrow \{p_n\}$ is conv., as it is a Cauchy sequence

(III) $\Rightarrow$ (IV)



Assume, M is not geodesically complete

$\Rightarrow \exists$ geodesic γ defined on (a,b) but not in b. Assume $\|\dot\gamma\| = 1$ so that $L(\gamma|_{[c,d]}) = d-c \quad \forall [c,d] \subset (a,b)$

Let $\{t_i\}$ be a sequence in (a,b), with $t_i \to b \Rightarrow \{t_i\}$ Cauchy sequence

$\Rightarrow d(\gamma(t_i), \gamma(t_j)) = |t_i - t_j| \Rightarrow \{\gamma(t_i)\}$ Cauchy sequence, so $\gamma(t_i) \to p_0$ by (III)

Let U be a nbhd of p_0 as in Lem 1.2.9, i.e $\exists \varepsilon > 0$ s.t. $B_\varepsilon(q)$ is a geod. ball s.t. $U \subset B_\varepsilon(q) \ \forall q \in U$

In particular, every geod. σ with $\sigma(0) \in U$ is def. at least on $(-\varepsilon, \varepsilon)$ and minimizing there

Let $b - \frac{\varepsilon}{2} < t_m < t_n < b$ s.t $\gamma(t_m), \gamma(t_n) \in U$ and

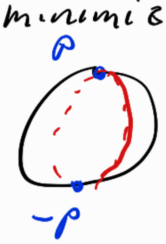
$\sigma(t) = \gamma(t - t_m)$, $t \in [0, t_n - t_m]$. $\sigma(0) = \gamma(t_m) \in U$

$\Rightarrow \sigma$ can be extended until $(-\varepsilon, \varepsilon) \Rightarrow \gamma$ extendable to $(a, t_0 + \varepsilon)$

As $b < t_m + \varepsilon$, this yields an extension of γ beyond b

(IV) $\Rightarrow$ (I) trivial

<u>Rem 1.3.5</u> (i) If (M,g) is complete we have $\forall p,q \in M \ \exists$ a minimising geodesic $\gamma \in \Omega_{p,q}$. Not nessecarily unique



(II) Compact Riem. mfds are always complete:

Every closed subset is again cpt $\Rightarrow$ (II) in Thm 1.3.2