1 Global Riemannian geometry

1.1 Riemannian manifolds as metric spaces

<u>Goal:</u> Equip a Riem mfd (M,g) (fixed for the rest of the subsec) with a distance fct (in the sense of metric spaces)

A continuous map $c:[a,b]\to M$ is called piecewise C^∞ $\iff \exists\, a=t_0<t_1<\ldots<t_k=b$ s.t. $c|_{[t_{i-1},t_i]}$ is C^∞ $\forall 1\leq i\leq k$

If $c:[a,b]\to M$ is p.w. C^∞, its length is def as

$$L(c)=\int_a^b \underbrace{\|\dot c(t)\|}_{=:\sqrt{\langle \dot c(t),\dot c(t)\rangle}}\,dt<\infty$$

<u>Lem 1.1.1</u> (i) $L(c)\geq 0$ & $L(c)=0 \iff c$ constant

(ii) $c_1:[a,b]\to M$, $c_2:[b,c]\to M$ p.w C^∞ with $c_1(b)=c_2(b)$ $\leadsto L(c_1\cup c_2)=L(c_1)+L(c_2)$

(iii) If $\varphi:[c,d]\to[a,b]$ is a diffeom. then $L(c\circ\varphi)=L(c)$

For $p,q\in M$, let $\Omega_{p,q}:=\{c\mid c$ p.w C^∞ from p to $q\}$ and define

$$d: M\times M\to\mathbb{R}$$
$$(p,q)\mapsto d(p,q):=\inf\{L(c)\mid c\in\Omega_{p,q}\}$$

<u>Thm 1.1.1</u> (i) (M,d) is a metric space

(ii) The topology induced by d is the same as the original topology of M

<u>Proof:</u> (i) <u>$d(p,q)=d(q,p)$</u>: Lem 1.1.1 (iii) ✓

<u>$d(p,q)\leq d(p,r)+d(r,q)$</u>: For $\varepsilon>0$, choose $c_1\in\Omega_{p,r}$, $c_2\in\Omega_{r,q}$ s.t. $L(c_1)\leq d(p,r)+\frac{\varepsilon}{2}$, $L(c_2)\leq d(r,q)+\varepsilon/2 \overset{\text{Lem 1.1.1}}{\underset{\text{(ii)}}{\leadsto}} L(\underbrace{c_1\cup c_2}_{\in\Omega_{p,q}})\leq d(p,r)+d(r,q)+\varepsilon$ ✓

$\underline{d(p,q)\geq 0,\ d(p,p)=0}$: Lem 1.1.1 (i) ✓

$\underline{d(p,q)=0 \Rightarrow p=q}$: Let $p\neq q\in M$, $\underline{(U,\varphi)\text{ a}}$ chart around p s.t. $\varphi(p)=0$. Choose $\varepsilon>0$ so small that $B_\varepsilon(0)\subset\varphi(U)\subset\mathbb{R}^n$ and

$q\notin K:=\varphi^{-1}(\overline{B_\varepsilon(0)})\subset M$

K cpt $\Rightarrow$ $L:=\{\sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\big|_r \mid r\in K, \sum_{i=1}^n a_i^2=1\}\subset TM$ also cpt.

$\Rightarrow f: TM\to\mathbb{R},\ v\mapsto g(v,v)=\|v\|^2$ has positive max & min on L

$\Rightarrow \exists R>1$ s.t. $R^2\geq\langle v,v\rangle\geq\frac{1}{R^2}\ \forall v\in L$

For general $v=\sum b_i\frac{\partial}{\partial x^i}\big|_r$, $r\in K$, $\frac{v}{\sqrt{\sum b_i^2}}\in L$

$$\Rightarrow\quad R^2\geq\frac{\langle v,v\rangle}{\sum_i b_i^2}\geq\frac{1}{R^2}\qquad(1.1.1)$$

Let $c\in\Omega_{p,q}$, then c there exists a smallest $s\in(a,b]$ s.t.

$c|_{[a,s]}\in K$ and $c(t)\in\partial K$

Can write $\varphi\circ c|_{[a,s]}=(c_1,\dots c_n):[a,s]\to\mathbb{R}^n \leadsto \dot c(t)=\sum_{i=1}^n\dot c_i(t)\frac{\partial}{\partial x^i}\big|_{c(t)}$

$$\Rightarrow L(c)\geq L(c|_{[a,s]})=\int_a^s\sqrt{\langle\dot c(t),\dot c(t)\rangle}\,dt$$

$$\overset{(1.1.1)}{\geq}\frac{1}{R}\int_a^s\sqrt{\sum_{i=1}^n(\dot c_i(t))^2}\,dt\qquad(1.1.2)$$

$$=\frac{1}{R}\underbrace{L(\varphi\circ c)}_{\text{w.r.t. }(\mathbb{R}^n,g_{eucl})}\geq\frac{1}{R}\|\varphi(c(s))\|\underset{c(s)\in\partial K}{=}\frac{\varepsilon}{K}$$

$$\Rightarrow\quad d(p,q)\geq\frac{\varepsilon}{R}>0\qquad(1.1.3)\quad ✓$$

(ii) By the above, we have for every chart (U,φ) an $r>0$ s.t.
$B_r(p)=\{q\in M\mid d(p,q)<r\}\subset U$ (take $r=\frac{\varepsilon}{R}$ in (1.1.3)))

On the other hand, $\forall r>0\ \exists V\subset M$ open s.t. $V\subset B_r(p)$:
Let $\delta<\min\{\varepsilon,\frac{r}{R}\}$ and set $V=\varphi^{-1}(B_\delta(0))$ (φ, R,ε as above)

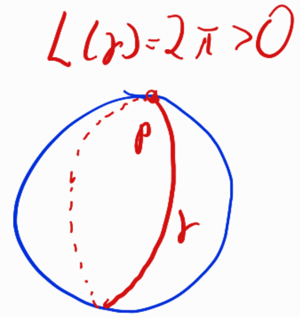
For $q \in V$ and $c(t) = \varphi^{-1}(t\varphi(q)) \in \Omega_{p,q}$,

$$L(c) = \int_0^1 \|\dot c(t)\|\, dt \overset{(1.1.1)}{\leq} \int_0^1 \sqrt{R^2(x^i(q))}\, dt \leq R\int_0^1 \|\varphi(q)\|\, dt < R\delta < r$$

$\Rightarrow d(p,q) < r \Rightarrow V \subset B_r(p)$ □

1.2 Geodesics are minimizing lengths

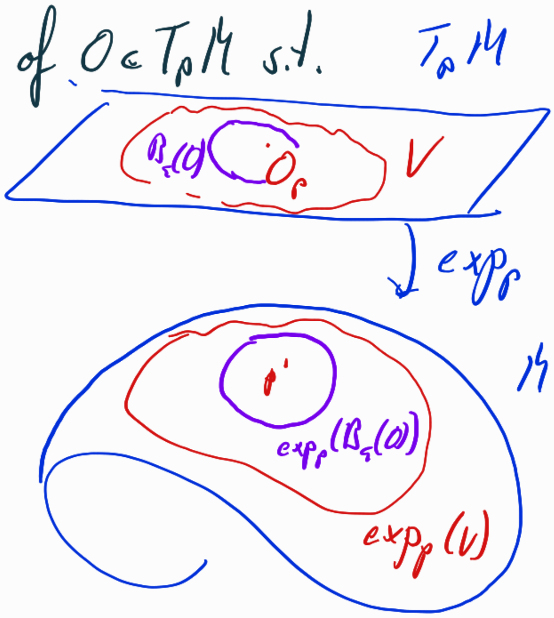
Fix again a Riem mfd (M,g). We want to show



Theorem 1.2.1 Let $p,q \in M$ and $\gamma \in \Omega_{p,q}$.
If $L(\gamma) = d(p,q)$ and $\|\dot\gamma\| \equiv \text{const}$, γ is a geodesic

Rem 1.2.2 Not every geodesic is length-minimizing, but we will show that every suff. short geod. is

Def 1.2.3 Let $p \in M$, $V \subset T_pM$ an open nbhd. of $0 \in T_pM$ s.t. $\exp_p|_V : V \to \exp_p(V)$ is a diffeo.
Then $\exp_p(V)$ is called normal nbhd. of p
For $\varepsilon > 0$ s.t. $B_\varepsilon(0) \subset V$, we call $\exp_p(B_\varepsilon(0))$ geodesic ball around p.



Lem 1.2.4 If $f : (a,b)\times(c,d) \to M$, $(t,s) \mapsto f(t,s)$ is C^∞, then

$$\frac{\nabla}{ds}\frac{\partial f}{\partial t} = \frac{\nabla}{dt}\frac{\partial f}{\partial s}$$

[Recall: $\frac{\partial f}{\partial t}(t,s) = T_{(t,s)}f(\partial_t) \in T_{f(t,s)}M$. For fixed t, $\frac{\partial f}{\partial t}$ is a vector field along $s \mapsto f(t,s)$ $\rightsquigarrow$ can build $\frac{\nabla}{ds}\frac{\partial f}{\partial t}$. Analogously $\frac{\nabla}{dt}\frac{\partial f}{\partial s}$]

Sketch of pf: In local coordinates, $\frac{\partial f}{\partial t} = \frac{\partial(x^i\circ f)}{\partial t}\partial_i\circ f$

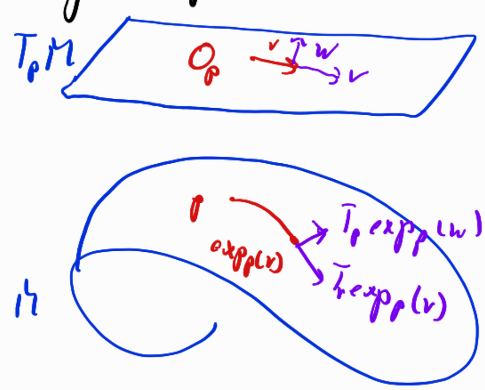
$$\frac{\nabla}{ds}\frac{\partial f}{\partial t} = \left(\frac{\partial^2(x^k\circ f)}{\partial s\,\partial t} + \frac{\partial(x^i\circ f)}{\partial s}\frac{\partial(x^j\circ f)}{\partial t}\Gamma^k_{ij}\circ f\right)\partial_k\circ f$$

$\frac{\nabla}{dt}\frac{\partial f}{\partial s}$ analogous. $\Gamma^k_{ij} = \Gamma^k_{ji}$ yields the result □

Lem 1.2.5 (Gauss-lemma) Let $p \in M$ and $\exp_p(v)$ be defined for $v \in T_pM$, and let $w \in T_v(T_pM) = T_pM$. Then,

$$\langle T_v \exp_p(v), T_w \exp_p(w) \rangle = \langle v, w \rangle$$

($\exp_p$ is a radial isometry)



Proof Let $\gamma(t) = \exp_p(tv) \Rightarrow \dot\gamma(1) = T_v \exp_p(v)$

For $w = \lambda \cdot v$, $v \in \mathbb{R}$,

$$\langle T_v \exp_p(v), T_v \exp_p(w) \rangle = \lambda \langle \dot\gamma(1), \dot\gamma(1) \rangle = \lambda \langle \dot\gamma(0), \dot\gamma(0) \rangle = \lambda \langle v, v \rangle = \langle v, w \rangle$$

$\Rightarrow$ Remains to prove the lemma for $v \perp w$:

Choose a curve $v(s)$ in T_pM s.t. $v(0) = v$, $\dot v(0) = w$ and $\|v(s)\| = \|v\|$

Choose $\delta, \varepsilon > 0$ s.t.

$$f : (-\delta, 1+\delta) \times (-\varepsilon, \varepsilon) \to M, \quad (t,s) \mapsto \exp_p(t v(s))$$

is well-defined

For s fixed, $t \mapsto f(t,s) = \exp_p(t v(s))$ is a geodesic and

$$\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle (1,0) = \langle (T_v \exp_p)(w), (T_v \exp_p)(v) \rangle \quad (1.1.4)$$

We have

$$\frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = \left\langle \frac{\nabla}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle + \left\langle \frac{\partial f}{\partial s}, \underbrace{\frac{\nabla}{dt} \frac{\partial f}{\partial t}}_{=0\ (t \mapsto f(t,s) \text{ geod})} \right\rangle$$

$$\text{Lem 1.2.4} \longrightarrow = \left\langle \frac{\nabla}{ds} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle$$

$$= \frac{1}{2} \frac{\partial}{\partial s} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle$$

$$= \frac{1}{2} \frac{\partial}{\partial s} \underbrace{\langle v(s), v(s) \rangle}_{\text{const. by ass.}} = 0$$

$\Rightarrow \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle$ indep. of t

We have $\lim\limits_{t \to 0} \frac{\partial f}{\partial s}(t,0) = \lim\limits_{t \to 0} T_{tv} \exp_p(tw) = 0$

$$\Rightarrow \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle (1,0) = 0 \overset{(1.1.4)}{\Longrightarrow} \text{result}$$

$\square$

Theorem 1.2.6 Let $p \in M$, $\mathcal{U} \subset M$ a normal nbhd of p, $B \subset \mathcal{U}$ a geodesic ball around p. Let $\gamma : [0,1] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma([0,1]) \subset B$. Then for every p.w C^∞ curve $c : [0,1] \to M$ with $c(0) = \gamma(0)$, $c(1) = \gamma(1)$, we have $L(\gamma) \leq L(c)$ and "=" $\Leftrightarrow$ c is a reparam. of γ.
In particular, $L(\gamma) = d(\gamma(0), \gamma(1))$

Proof Let $w = \dot\gamma(0)$. As $\|\dot\gamma\| \equiv$ const, $L(\gamma) = \int_0^1 \|\dot\gamma(t)\| dt = \int_0^1 \|w\| dt = \|w\|$

Let c be as in the thm. We may assume $c(t) \neq p$ for $t > 0$
(Otherwise, restrict to the shorter curve $c|_{[t_0, 1]}$, where $t_0 = \max\{t \in [0,1] \mid c(t) = p\}$)

Assume for the moment that $c([0,1]) \subset B$
$\exp_p$ diffeo onto $\mathcal{U}$ $\Rightarrow$ write $c(t) = \exp_p(r(t) \cdot v(t))$ for $t > 0$, where
$r : (0,1] \to \mathbb{R}_+$, $v : (0,1] \to T_pM$, with $\|v(t)\| = 1$ are p.w. C^∞

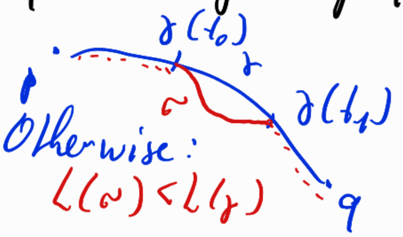
Wherever r, v are C^∞,
$$\tfrac{d}{dt} r(t) \cdot v(t) = \dot r(t) v(t) + r(t) \dot v(t) \quad (1.1.5),$$
and $\langle \dot v(t), v(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle v(t), v(t) \rangle = 0$
$$\Rightarrow \|\dot c\|^2 = \langle T_{rv}(\dot r v + r \dot v), T_{rv} \exp_p(\dot r v + r v) \rangle$$
$$\text{Lem 1.2.5} \leadsto = (\dot r)^2 + \|T_{rv} \exp_p(r \dot v)\|^2 \geq (\dot r)^2$$
$$\Rightarrow L(c) = \int_0^1 \|\dot c(t)\| dt \geq \lim_{\varepsilon \to 0} \int_\varepsilon^1 |\dot r(t)| dt$$
$$\geq \lim_{\varepsilon \to 0} \int_\varepsilon^1 \dot r(t) dt = \lim_{\varepsilon \to 0} r(1) - \underbrace{r(\varepsilon)}_{\to 0} = \|w\| = L(\gamma)$$
$\Rightarrow L(c) \geq L(\gamma)$ and "=" holds iff $\dot v \equiv 0$ and $\dot r \geq 0$
$\Rightarrow c(t) = \exp_p(r(t) v) = \gamma(r(t))$ is a rep. of γ

If c leaves B, then $\exists t_0 \in (0,1)$ s.t. $c(t_0) \in \partial B = \exp_p(\partial B_\delta(0))$ for some $\delta > 0$ and along the above arguments,
$$L(c) \geq L(c|_{[0,t_0]}) \geq \delta > \|w\| = L(\gamma)$$
$\square$

Corollary 1.2.7 (i) Every suff. short geodesic is minimizing:
If $\gamma : [0,1] \to M$ is a geodesic, $\exists \varepsilon > 0$ s.t. $L(\gamma|_{[0,t]}) = d(\gamma(0), \gamma(t))$ $\forall t \in (0,\varepsilon)$
(ii) If $\exp_p(B_\delta(0))$ is a geodesic ball, we have
$$\exp_p(B_\delta(0)) = B_\delta(p) := \{q \in M \mid d(p,q) < \delta\}$$

Pf: (i) Take $\varepsilon > 0$ s.t. $\gamma([0,\varepsilon])$ is in a geod. ball and apply Thm 1.2.6
(ii) "$\subseteq$" If $q \in \exp_p(B_\delta(0))$, then $q = \exp_p(v)$ for some $v \in T_pM$ with $\|v\| < \delta$
and the geod. $\gamma : [0,1] \ni t \mapsto \exp_p(tv)$ joins p & q with $d(p,q) = L(\gamma) = \|v\| < \delta$
"$\supseteq$" If $q \notin \exp_p(B_\delta(0))$, then every $c \in \Omega_{p,q}$ has $L(c) \geq \delta$, see pf. of Thm 1.2.6 $\blacksquare$

Pf of Thm 1.2.1 Let $\gamma : [0,1] \to M$ be in $\Omega_{p,q}$ and $L(\gamma) = d(p,q)$
Then $\forall t_1 < t_2 \in [0,1]$, $L(\gamma|_{[t_1,t_2]}) = d(\gamma(t_1), \gamma(t_2))$



For t_1 arbitrary and t_2 suff close to t_1,
$\gamma|_{[t_1,t_2]}$ lies in a geodesic ball around $\gamma(t_1)$,
hence is a reparam. of a geod. by Thm 1.2.6
But $\|\dot\gamma\| \equiv$ const and thus, $\gamma|_{[t_1,t_2]}$ is a geodesic itself [DG, VT23]
t_1 arbitrary $\Rightarrow$ γ is a geod. $\square$

Lem 1.2.8 Consider the map $F : TM \to M \times M$, $v \mapsto (\pi(v), \exp_{\pi(v)}(v))$
(Recall: $TM = \bigsqcup_{p \in M} T_pM$ and $\pi : TM \to M$ is the canonical projection)

Then, $T_{0_p}F : T_{0_p}TM \to T_{(p,p)}(M \times M)$ is an isomorphism $\forall p \in M$

Pf We have canonical decompositions
$$T_{0_p}TM \cong \underbrace{T_pM}_{\cong\, T_{0_p}(\{0_q \mid q \in M\})} \oplus \underbrace{T_pM}_{=\, T_{0_p}(T_pM)} \qquad \text{and} \qquad T_{(p,p)}(M \times M) \cong T_pM \oplus T_pM$$
where $\{0_q \mid q \in M\} \subset TM$ and $T_pM \subset TM$.

$\Rightarrow$ For a curve $0_{c(t)} \in \{0_q \mid q \in M\}$,
$$F(0_{c(t)}) = (\pi(0_{c(t)}), \exp_{c(t)}(0_{c(t)})) = (c(t), c(t))$$
$$\Rightarrow T_{0_p}F(\dot c(t), 0) = (\dot c(t), \dot c(t))$$

For a curve $c(t)\in T_pM\subset TM$,

$$F(c(t)) = (\pi(c(t)), \exp_{\pi(c(t))}(c(t))) = (p, \exp_p(c(t)))$$

$$\Rightarrow T_{0_p}F(0,\dot c(t)) = (0, T_{0_p}\exp_p(\dot c(t)))$$

$$\Rightarrow T_{0_p}F = \begin{pmatrix} id & 0 \\ id & \underbrace{T_{0_p}\exp_p}_{=id} \end{pmatrix} \Rightarrow T_{0_p}F \text{ iso}$$

□

<u>Lem 1.2.9</u> $\forall p\in M$, there exists an open nbhd U of p & $\varepsilon>0$ such that for all $q\in U$, we have:

(i) $\exp_q|_{B_\varepsilon(0)}$ is a diffeo onto its image

(ii) $U\subset \exp_q(B_\varepsilon(0)) = B_\varepsilon(q) \Rightarrow U$ is a normal nbhd of each of its pts

(iii) For all $q_1,q_2\in U$, there exists a unique minimizing geodesic $\gamma\in\Omega_{q_1,q_2}$, $\gamma:[0,1]\to M$ with $L(\gamma)=d(p,q)<\varepsilon$

<u>Pf</u> By Lem 1.2.8, there exist nbhds $U'\subset TM$ of 0_p and $W'\subset M\times M$ of (p,p) st. $F|_{U'}:U'\to W'$ is a diffeom. We may assume

$$U' = \{v\in T_qM \mid q\in V', \|v\|<\varepsilon\}$$

for an open nbhd $V'\subset M$ of p and $\varepsilon>0$.

Choose a nbhd U of p s.t. $U\times U\subset W' \Rightarrow U\subset V'$ ($F^{-1}(q,q)=0_q\in U' \Rightarrow q\in V'$)

(i) $\forall q\in U\subset V'$, $B_\varepsilon(0)=U'\cap T_qM$ and $\exp_q|_{B_\varepsilon(0)}$ is a diffeo as $F|_{U'\cap T_qM}: v\mapsto (q,\exp_q(v))$ is

(ii) By Cor 1.2.7, $\exp_q(B_\varepsilon(0)) = B_\varepsilon(q)$ $\forall q\in U$

For $q,x\in U$, we have $(q,x)=F(v)=(\pi(v),\exp_{\pi(v)}(v))$ for $v\in U'$

$\Rightarrow \pi(v)=q \Rightarrow v\in T_qM$ and $\|v\|<\varepsilon$ by the choice of $U' \Rightarrow x\in B_\varepsilon(q)$

$\Rightarrow U\subset B_\varepsilon(q)$

(iii) $q_2=\exp_{q_1}(v)$ for some $v\in T_{q_1}M$, $\|v\|<\varepsilon$ by (ii)

$\Rightarrow \gamma(t)=\exp_{q_1}(tv)$ does the job. Every other geodesic $\exp_{q_1}(tw)$ can not hit q_2 if $\|tw\|<\varepsilon$ ($\exp_{q_1}$ diffeo) □