1.4 Jacobi fields and konjugate points

Goal: Relate curvature to spreading of geodesics

Let $c:[a,b]\to M$ be a C^∞ curve. A C^∞ map $f:(-\varepsilon,\varepsilon)\times[a,b]\to M$, $(s,t)\mapsto f(s,t)$
is called variation of c, if $f|_{s=0}=c$
We call $\frac{\partial f}{\partial s}|_{s=0}$ the variational vector field of f
If $c=\gamma$ is a geodesic, we call f geodesic variation,
if $t\mapsto f(s,t)$ is a geodesic for all $s\in(-\varepsilon,\varepsilon)$

Lem 1.4.1 For $V\in C^\infty(I,TM)$ s.t. $V(s,t)\in T_{f(s,t)}M$ (with f as above)
we have $\frac{\nabla}{ds}\frac{\nabla}{dt}V-\frac{\nabla}{dt}\frac{\nabla}{ds}V=R(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t})V$

Pf: Long calculation in local coordinates □

Thm 1.4.2 Let γ be a geodesic and f be a geodesic variation of γ.
Then its variational vector field $J:=\frac{\partial f}{\partial s}|_{s=0}$ satisfies
$$\frac{\nabla^2}{dt^2}J+R(J,\dot\gamma)\dot\gamma=0 \qquad (1.4.1)$$

Pf: $t\mapsto f(s,t)$ geodesic $\Rightarrow \frac{\nabla}{dt}\frac{\partial f}{\partial t}=0$

$$\Rightarrow 0=\frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial f}{\partial t}\overset{\text{Lem 1.4.1}}{=}\frac{\nabla}{dt}\underbrace{\frac{\nabla}{ds}\frac{\partial f}{\partial t}}_{=\frac{\nabla}{dt}\frac{\partial f}{\partial s}\ (\text{Lem 1.2.4})}+R(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t})\frac{\partial f}{\partial t}$$

Restrict to $s=0\Rightarrow(1.4.1)$ □

Def 1.4.3 Let γ be a geodesic. A VF J along γ is called Jacobi field,
if it satisfies (1.4.1) (which is called the Jacobi equation)

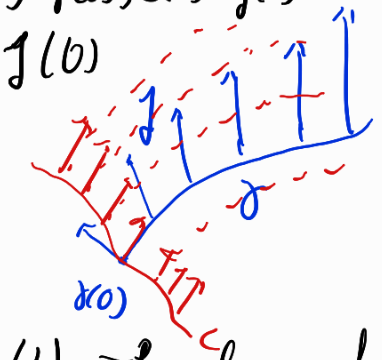
Thm 1.4.4 Let $\gamma:[a,b]\to M$ be a geodesic. Then for $v,w\in T_{\gamma(a)}M$, there exists
a unique Jacobi field J along γ s.t. $J(a)=v$, $\frac{\nabla}{dt}J(a)=w$

Pf: Let $\{v_i\}_{1\le i\le n}$ be an ONB of $T_{\gamma(a)}M$ and $\{E_i\}_{1\le i\le n}$ be parallel VFs along γ s.t.
$E_i(0)=v_i$ $\leadsto$ $\{E_i\}_{1\le i\le n}$ is an ONB for all $T_{\gamma(t)}M$. Write $J(t)=\sum_i f_i(t)\cdot E_i(t)$
Then (1.4.1) is equivalent to
$$f_i''+\sum_j a_{ij}f_j=0\ \ \forall i,\quad a_{ij}:=\langle R(E_j,\dot\gamma)\dot\gamma,E_i\rangle$$
and the result follows from ODE theory □

Thm 1.4.5 Let $\gamma:[a,b]\to M$ be a geodesic and J a Jacobi field along γ, then there exists a geodesic variation f of γ with variational vector field J

Pf Assume $a=0$. Let $c:(-\varepsilon,\varepsilon)\to M$ be C^∞ and s.t. $c(0)=\gamma(0)$, $\dot c(0)=J(0)$

Let X be a VF along c s.t. $X(0)=\dot\gamma(0)$ and $\frac{\nabla}{ds}X(0)=\frac{\nabla}{dt}J(0)$



(e.g. $X(s)=Y(s)+sZ(s)$, Y,Z parallel with $Y(0)=\dot\gamma(0)$, $Z(0)=\frac{\nabla}{dt}J(0)$)

Set $\quad f(s,t)=\exp_{c(s)}(t\,X(s))$

$t\mapsto f(s,t)$ is a geod. and $f(0,t)=\exp_{\gamma(0)}(t\dot\gamma(0))=\gamma(t) \Rightarrow f$ geod. var. of γ

$\overset{\text{Thm 1.4.2}}{\Rightarrow}$ $I:=\frac{\partial f}{\partial s}\big|_{s=0}$ is a Jacobi field and

$I(0)=\frac{\partial f}{\partial s}(0,0)=\dot c(0)=J(0)$

$\frac{\nabla}{dt}I(0)=\frac{\nabla}{dt}\frac{\partial f}{\partial s}(0,0)\underset{1.2.4}{\overset{\text{Lem}}{=}}\frac{\nabla}{ds}\frac{\partial f}{\partial t}(0,0)=\frac{\nabla}{ds}X(0)=\frac{\nabla}{dt}J(0)\underset{1.4.4}{\overset{\text{Thm}}{\Rightarrow}} I=J$ $\square$

Corollary 1.4.6 If $\gamma:[0,b]\to M$ is a geod., J a Jacobi field along γ, $J(0)=0$

Then, $J(t)=T_{tv}\exp_p(tw)$, where $v=\dot\gamma(0)$, $w=\frac{\nabla}{dt}J(0)$

Pf Take $c\equiv\gamma(0)$ and $X(s)=v+sw$ in the pf of Thm 1.4.5 s.t.

$f(s,t)=\exp_p(t(v+sw))$

$\Rightarrow \frac{\partial f}{\partial s}(0,t)=T_{tv}\exp_p(tw)$ $\square$

Lem 1.4.7 Let $\gamma:[0,b]\to M$ be a geodesic and J a Jacobi field along γ

(i) $J^{\parallel}:=\frac{\langle J,\dot\gamma\rangle}{\langle\dot\gamma,\dot\gamma\rangle}\dot\gamma$ and $J^{\perp}:=J-J^{\parallel}$ are also Jacobi fields

(ii) $J\perp\dot\gamma\ \forall t\in[0,b] \Leftrightarrow J(0)\perp\dot\gamma(0)$ & $\frac{\nabla}{dt}J(0)\perp\dot\gamma(0)$

(iii) $J\parallel\dot\gamma\ \forall t\in[0,b]\Leftrightarrow J(0)=\alpha\dot\gamma(0)$ & $\frac{\nabla}{dt}J(0)=\beta\dot\gamma(0)$ for some $\alpha,\beta\in\mathbb{R}$

In this case, $J(t)=(\alpha+t\beta)\dot\gamma(t)$

Pf Assume for simplicity $\|\dot\gamma\|=1$. Since J is a Jacobi field,

$\frac{\nabla^2}{dt^2}\langle J,\dot\gamma\rangle\dot\gamma=\langle\frac{\nabla^2}{dt^2}J,\dot\gamma\rangle\dot\gamma=-\underbrace{\langle R(J,\dot\gamma)\dot\gamma,\dot\gamma\rangle}_{=0}\dot\gamma=-\underbrace{R(\langle J,\dot\gamma\rangle\dot\gamma,\dot\gamma)\dot\gamma}_{=0}$

$\Rightarrow J^{\parallel}$ JF $\overset{\text{lin.}}{\Rightarrow} J^{\perp}=J-J^{\parallel}$ JF $\Rightarrow$ (i)

We have seen $\frac{d^2}{dt^2}\langle J,\dot\gamma\rangle=0\Rightarrow\langle J,\dot\gamma\rangle=\alpha+t\beta$ with $\alpha=\langle J(0),\dot\gamma(0)\rangle$,

$\beta=\frac{d}{dt}\langle J,\dot\gamma\rangle|_{t=0}=\langle\frac{\nabla}{dt}J(0),\dot\gamma(0)\rangle \Rightarrow$ (ii) + (iii)

Rem 1.4.7 The space $\mathcal{J}_\gamma = \{J \mid J \text{ JF along } \gamma\}$ splits as

$$\mathcal{J}_\gamma = \underbrace{\mathcal{J}_\gamma^\perp}_{\substack{\text{normal} \\ \text{JFds} \\ 2n-2\text{-dim}}} \oplus \underbrace{\mathcal{J}_\gamma^{\parallel}}_{\substack{\text{parallel} \\ \text{JFds} \\ 2\text{-dim}}} \quad \text{if } \dim M = n$$

Parallel Jacobi fields are called trivial (they correspond to reparam. of γ), normal Jacobi fields are called essential

Thm 1.4.8 Let (M,g) be a Riem. mfd of constant curvature $\varkappa$, $\gamma:[0,a]\to M$ a geodesic with $\|\dot\gamma\|=1$ and J an essential Jacobi field along γ with $J(0)=0$.

$$\text{Then,} \quad J(t) = \begin{cases} \dfrac{\sin(t\sqrt{\varkappa})}{\sqrt{\varkappa}} \cdot X(t), & \varkappa > 0, \\ t \cdot X(t) & \varkappa = 0, \\ \dfrac{\sinh(t\sqrt{-\varkappa})}{\sqrt{-\varkappa}} X(t) & \varkappa < 0, \end{cases} \qquad (1.4.2)$$

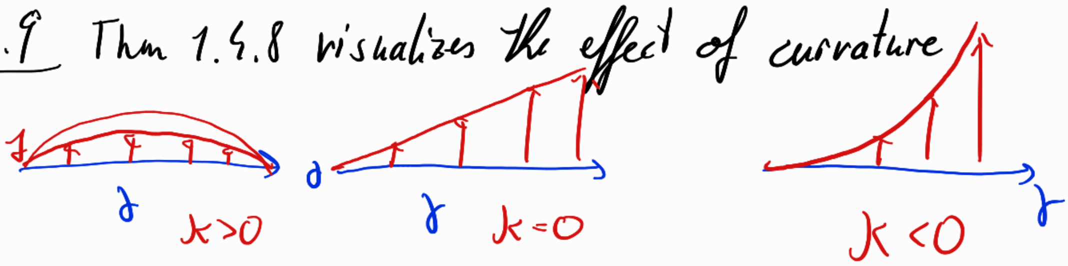
where $X(t)$ is the parallel vf along γ with $X(0) = \frac{D}{dt}J(0)$.

Pf constant curvature $\varkappa \Rightarrow R(J,\dot\gamma)\dot\gamma = \varkappa(\langle\dot\gamma,\dot\gamma\rangle J - \underbrace{\langle J,\dot\gamma\rangle}_{=0 \text{ (essential JF)}}\dot\gamma)$

(1.4.1) is $\frac{D^2}{dt^2}J + \varkappa J = 0$

$\Rightarrow$ Check that (1.4.2) satisfies (1.4.2) with the right initial condition.

Rem 1.4.9 Thm 1.4.8 visualizes the effect of curvature



Def 1.4.10 Let $\gamma:[a,b]\to M$ be a nonconstant geodesic and $p=\gamma(a)$, $q=\gamma(b)$. q is called conjugate to p along γ if there is a Jacobi field $J\neq 0$ along γ s.t. $J(a)=0$, $J(b)=0$. We call the dimension of

$$\{J \mid J \text{ JF along } \gamma, J(a)=0, J(b)=0\}$$

multiplicity of the conjugate point. It can be at most $n-1$, where $n=\dim(M)$: We have $J(a)=0$ and $J\perp\gamma'$ (check!), so $\frac{D}{dt}J(a)\perp\dot\gamma(a)$ and these determine J completely

Ex 1.4.11 (i) (S^n, g_{S^n}) has const. curvature 1. Essential vfds along a geod. $\gamma:[0,a]\to M$ with $J(0)=0$ are all of the form $J(t)=\sin(t)\cdot X$, with X parallel and $X\perp\dot\gamma$

$\Rightarrow$ $\gamma(\pi)=-\gamma(0)$ is conj. to $\gamma(0)$ with mult. $n-1$.

(ii) If (M,g) is of constant curvature $\kappa\leq 0$, there are no conjugate points

Thm 1.4.12 Let $\gamma:[0,a]\to M$ be a geodesic, $p=\gamma(0)$, $v=\dot\gamma(0)$ and $t_0\in[0,a]$. Then:
$\gamma(t_0)$ is conj. to p along γ $\Longleftrightarrow$ $t_0 v\in T_pM$ is a critical point of $\exp_p$

In that case, its multiplicity equals the dimension of $\ker(T_{t_0 v}\exp_p)$

Pf: Let J be a vf along γ with $J(0)=0$. By Cor. 1.4.6,

J is a JF $\Longleftrightarrow$ $J(t)=T_{tv}\exp_p(tw)$

In that case $J\neq 0\Leftrightarrow w\neq 0$ and $J(t_0)=0\Longleftrightarrow w\in\ker T_{t_0v}\exp_p$ □

Corollary 1.4.13 Let $\gamma:[0,a]\to M$ be a geodesic. Then $\varepsilon>0$ s.t. $\forall t\in(0,\varepsilon)$, $\gamma(t)$ is not conj. to $\gamma(0)$ along $\gamma|_{[0,t]}$

Pf Choose $\varepsilon>0$ s.t. $\gamma([0,\varepsilon])$ lies in a normal nbhd of p

$\Rightarrow$ $T_{t\dot\gamma(0)}\exp_p$ isomorphism. $\forall t\in(0,\varepsilon)$ $\Rightarrow$ result follows from Thm 1.4.12 □

1.5 Fundamental group and covering maps

Def 1.5.1 Let X be a top space and $p\in X$.

Let $\Omega_p:=\{c:[0,1]\to X\mid c \text{ cont.}, c(0)=c(1)=p\}$

We say two paths $c_0, c_1\in\Omega_p$ are homotopic $(c_0\sim c_1)$ if they can be connected by a path $[0,1]\ni s\mapsto c_s$ in Ω_p s.t. $(s,t)\mapsto c_s(t)$ is continuous

$\pi_1(X,p)=\Omega_p/\sim$ is called fundamental group, where the group structure is def by $[c_0]*[c_1]=[c_0\cup c_1]$, where $(c_0\cup c_1)(t)=\begin{cases} c_0(2t), & t\in[0,\frac12] \\ c_1(2t-1), & t\in[\frac12,1]\end{cases}$

Rem 1.5.2 If X is path connected and $p,q\in X$,

$\pi_1(X,p)\cong\pi_1(X,q)$ $\rightsquigarrow$ Write $\pi_1(X)$

Def 1.5.3 If X is connected, we call X simply connected, if $\pi_1(X)=\{0\}$

Ex 1.5.4 (i) $\pi_1(\mathbb{R}^n) = \{0\}$, $\pi_1(S^n) = \{0\}$ for $n \geq 2$

(ii) $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(\underbrace{T^n}_{= S^1 \times \dots \times S^1 \text{ (n times)}}) = \mathbb{Z}^n$

Def 1.5.5 Let X, Y be top. spaces, a continuous map $\pi : X \to Y$ is called covering map if $\forall y \in Y$, there is an open nbhd U of y s.t.

$$\pi^{-1}(U) = \bigsqcup_{i \in I} U_i,$$

where $U_i \subset X$ are open and $\pi|_{U_i} : U_i \to U$ is a homeom. $\forall i \in I$

If X, Y are C^∞ mfds and $\pi|_{U_i} : U_i \to U$ a diffeo $\forall i \in I$, we call π C^∞ covering

— " — Riem mfds — " — an isometry — " — Riem. covering

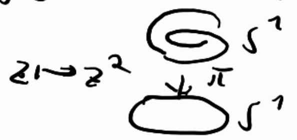
If $\pi_1(X) = \{0\}$, we call π universal covering.

Rem 1.5.6 (i) $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$ are covering maps, there exists a homeo/diffeo/isometry $\varphi : X_1 \to X_2$ s.t. $\pi_2 \circ \varphi = \pi_1$. In this sense, the universal cover is unique. We often write $\pi : \tilde{Y} \to Y$ for it.

There is a bijection between $\pi_1(Y)$ and $\pi^{-1}(y) \subset \tilde{Y}$ $\forall y \in Y$

(ii) Every connected mfd M admits a universal cover $\pi : \tilde{M} \to M$. If (M, g) is a connected Riem. mfd, we can equip $\tilde{M}$ with the metric $\tilde{g} := \pi^* g$ and π becomes a local isometry

Examples 1.5.7 (i) For every $k \in \mathbb{Z}$, $\pi : S^1 \to S^1$, $z \mapsto z^k$ is a covering map



(ii) $\pi : \mathbb{R} \to S^1$, $x \mapsto e^{ix}$ is (the) universal cover of S^1. Observe: $\pi^{-1}(e^{ix}) = \{x + 2\pi k \mid k \in \mathbb{Z}\}$

(iii) $\pi : S^n \to \mathbb{R}P^n$, $p \mapsto [p] = \pm p$ is the universal cover of $\mathbb{R}P^n$, $n \geq 2$.

$\Rightarrow$ $\pi_1(\mathbb{R}P^n) = \mathbb{Z}^2$

(iv) $\pi : \mathbb{R}^n \to T^n$, $(x_1, \dots, x_n) \mapsto (e^{ix_1}, \dots, e^{ix_n})$ is the universal cover of T^n