1.6 Nonpositive curvature: The Cartan Hadamard theorem

<u>Theorem 1.6.1</u> Let M be a complete Riem. mfd with sectional curvature $K \leq 0$. Then M does not contain conjugate points. In particular, $\exp_p : T_pM \to M$ is $\forall p \in M$ a local diffeom. around each $v \in T_pM$

<u>Pf</u> Let $\gamma : \mathbb{R} \to M$ be a nonconstant geodesic, $J \neq 0$ a Jacobi ield along γ with $J(0)$

Let $f(t) = \|J(t)\|^2$. Then, $f(0) = 0$, f

$$f'(0) = 2\langle \tfrac{\nabla}{dt} J(0), J(0) \rangle = 0$$

$$f''(0) = 2\langle \tfrac{\nabla}{dt} J(0), \tfrac{\nabla}{dt} J(0) \rangle > 0 \text{ (otherwise } \tfrac{\nabla}{dt} J(0) = 0 \Rightarrow J \equiv 0 \text{)}$$

$$f = \langle J, J \rangle'' = \underbrace{2\langle \tfrac{\nabla J}{dt}, \tfrac{\nabla J}{dt} \rangle}_{\geq 0} + \underbrace{2\langle \tfrac{\nabla^2}{dt^2} J, J \rangle}_{-\langle R(J,\dot\gamma)\dot\gamma, J \rangle = -\underbrace{K(\operatorname{span}\{J,\dot\gamma\})}_{\leq 0}\underbrace{(\langle J,J\rangle\langle\dot\gamma,\dot\gamma\rangle - \langle J,\dot\gamma\rangle^2)}_{\geq 0 \text{ (CS-ineq)}} \geq 0} \geq 0$$

$\Rightarrow f'(t) > 0 \ \forall t > 0 \Rightarrow f(t) > 0 \ \forall t > 0$

$\Rightarrow$ No conj. pts! $\overset{\text{Thm}}{\underset{1.4.12}{\Rightarrow}}$ $T_v \exp_p$ is an isom. $\forall v \in T_pM$ $\overset{\text{inverse}}{\underset{\text{fct thm}}{\Rightarrow}}$ $\exp_p$ is a loc. diffeom. $\square$

<u>Thm 1.6.2</u> Let M, N be Riem. mfds, M connected & complete and $\pi : M \to N$ a surj. local isometry. Then N is connected & complete and π is a covering map.

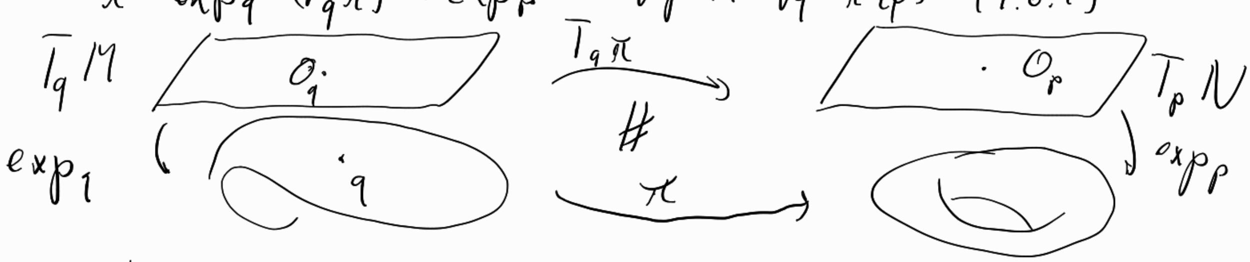
<u>Proof</u>: M conn & π surj. $\Rightarrow N = \pi(M)$ connected

<u>N complete</u>: Let $p \in N$ and $v \in T_pN$. Choose $q \in \pi^{-1}(p)$ and let $w = (T_q\pi)^{-1}(v) \in T_qM$

M complete $\Rightarrow$ The geod. $\gamma(t) = \exp_q(tw)$ is def $\forall t \in \mathbb{R}$. The curve $\pi \circ \gamma(t)$ is a geodesic (because π is a local isometry $\sim$ [DG 1]) with initial data $\pi(\gamma(0)) = \pi(q) = p$ and $(\pi \circ \gamma)^\circ(0) = T_{\gamma(0)}\pi(\dot\gamma(0)) = T_q\pi(w) = v$, defined $\forall t \in \mathbb{R} \Rightarrow N$ complete (p, v arbitrary) ✓

<u>π covering map</u>: Note that we have shown $\pi \circ \exp_q(tw) = \exp_p(tv)$, i.e

$$\pi \circ \exp_q \circ (T_q\pi)^{-1} = \exp_p \quad \forall p \in M \ \forall q \in \pi^{-1}(p) \quad (1.6.1)$$



Fix $p \in N$ and let $\varepsilon > 0$ be so small that $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p)$ is a diffeom.

<u>Claim!</u>: $\pi(B_\varepsilon(q)) = B_\varepsilon(p)$ and $\pi : B_\varepsilon(q) \to B_\varepsilon(p)$ is an isometry $\forall q \in \pi^{-1}(p)$

By (1.6.1), the map
$$\psi := \exp_q \circ (T_q\pi)^{-1} \circ (\exp_p|_{B_\varepsilon(0)})^{-1} : B_\varepsilon(p) \to M$$
satisfies $\pi \circ \psi = id_{B_\varepsilon(p)}$ and

$T_q\pi$ isom, M complete

$$\psi(B_\varepsilon(p)) = \exp_q((T_q\pi)^{-1}(B_\varepsilon(0))) = \exp_q(B_\varepsilon(0)) = B_\varepsilon(q)$$
$\Rightarrow \pi|_{B_\varepsilon(q)} : B_\varepsilon(q) \to B_\varepsilon(p)$ is a bijective local isometry, hence isometry

Claim 2 $\pi^{-1}(B_\varepsilon(p)) = \bigcup_{q\in\pi^{-1}(p)} B_\varepsilon(q)$

"$\supseteq$" If $q' \in B_\varepsilon(q)$ for some $q \in \pi^{-1}(p)$, then $q' = \psi(p')$ for some $p' \in B_\varepsilon(p)$ with ψ as above and $\pi(q') = p' \in B_\varepsilon(p)$

"$\subseteq$" Let $q' \in \pi^{-1}(B_\varepsilon(p))$ and $p' = \pi(q') \in B_\varepsilon(p)$.
We have a geodesic σ in $B_\varepsilon(p)$ from p' to p of $L(\sigma) < \varepsilon$ which can be written as $\sigma(t) = \exp_{p'}(tv')$, $t \in [0,1]$.
Let $w' = (T_{q'}\pi)^{-1}(v')$ and consider the geodesic $\tau(t) = \exp_{q'}(tw')$ connecting $q' = \tau(0)$ and $q := \tau(1)$.
Because π is a local isometry, $\pi \circ \tau = \sigma \Rightarrow q = \pi^{-1}(p)$ and $L(\tau) = L(\sigma) < \varepsilon \Rightarrow q' \in B_\varepsilon(q)$

Claim 3 $\pi^{-1}(B_{\varepsilon/2}(p)) = \bigsqcup_{q\in\pi^{-1}(p)} B_{\varepsilon/2}(q)$ and $\pi : B_{\varepsilon/2}(q) \to B_{\varepsilon/2}(q)$ is an isometry $\forall q \in \pi^{-1}(p)$

Pf We only need to show that all $B_{\varepsilon/2}(q)$ are pairwise disj.
If $r \in B_{\varepsilon/2}(q_1) \cap B_{\varepsilon/2}(q_2)$ for $q_1, q_2 \in \pi^{-1}(p)$, then $q_2 \in B_\varepsilon(q_1)$ and $\pi(q_1) = \pi(q_2) = p \Rightarrow \lightning$ $\pi : B_\varepsilon(q_1) \to B_\varepsilon(p)$ being bijective

$\square$

Thm 1.6.3 (Cartan-Hadamard) Let M be a complete connected Riem. mfd of sectional curvature $K \leq 0$. Then $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map. In particular, if M is simply connected, it is diffeom. to $\mathbb{R}^n$

Pf M complete $\Rightarrow \exp_p : T_pM \to M$ is surjective (Thm 1.3.2)

$K \leq 0 \Rightarrow$ — " — is a local diffeo (Thm 1.6.1)

Let g be the metric on M, and equip T_pM with the metric $\hat{g} := \exp_p^* g$

$\leadsto \exp_p : T_pM \to M$ becomes a local isometry

$\forall v \in T_pM, \mathbb{R} \ni t \mapsto tv$ is a geodesic in T_pM, as they map to geod. on M under the loc. isometry $\exp_p$ $\Rightarrow$ The geod. starting in $0 \in T_pM$ are def. $\forall t \in \mathbb{R}$

Thm 1.3.2 $\Rightarrow (T_pM, \hat{g})$ is complete $\overset{\text{Prop}}{\underset{1.6.2}{\Rightarrow}}$ $\exp_p$ is a covering map.

T_pM simply connected $\Rightarrow \exp_p : T_pM \to M$ is the universal cover, hence a diffeo if M is simply connected as well. $\square$

<u>Rem 1.6.4</u> Essence of Thm 1.6.3: All topological information on complete Riem mfds of $K \leq 0$ is contained is contained in their fundamental group.

<u>Ex 1.6.5</u> $S^n, n \geq 2$ can not be equipped with a metric of sectional curvature $K \leq 0$

1.7 Positive curvature: The Bonnet-Myers theorem

Fix a Riem. mfd M. For $p, q \in M$, let

$$\Omega_{p,q} := \{ c : [0,1] \to M \mid c \ C^\infty, c(0) = p, c(1) = q \}$$

We would like to study the variation of the length functional $L : \Omega_{p,q} \to \mathbb{R}$, $c \mapsto \int_0^1 \|\dot{c}(t)\| dt$

However, $\|.\|$ is not C^∞ in 0. We therefore consider instead the energy

$$E : \Omega_{p,q} \to \mathbb{R}, \quad c \mapsto \int_0^1 \|\dot{c}(t)\|^2 dt$$

<u>Lem 1.7.1</u> (i) For any $c \in \Omega_{p,q}$, we have $L(c) \leq E^2(c)$ and "=" $\Leftrightarrow \|\dot{c}\|$ constant

(ii) If $\gamma \in \Omega_{p,q}$ is a minimizing geodesic, then $E(\gamma) \leq E(c)$ $\forall c \in \Omega_{p,q}$ and "=" $\Leftrightarrow$ c is another minimizing geodesic

<u>Pf</u> (i) CS-ineq: $L(c) = \int_0^1 1 \cdot \|\dot{c}(t)\| dt \leq \left(\int_0^1 1^2 dt\right)^{1/2} \left(\int_0^1 \|\dot{c}(t)\|^2 dt\right)^{1/2} = (E(c))^{1/2}$

and "=" $\Leftrightarrow \|\dot{c}(t)\|$ const

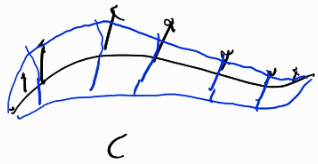
(ii) γ geod. $\Rightarrow \|\dot{\gamma}\|$ const $\Rightarrow E^2(\gamma) = L(\gamma) \leq L(c) \overset{(i)}{\leq} E(c)$ and "="

$\Leftrightarrow L(c) = L(\gamma)$ and (by (i)) $\|\dot{c}\| \equiv$ const $\Rightarrow c$ min. geodesic $\square$

A variation f of $c \in \Omega_{p,q}$ is called proper, if $f(s,.) \in \Omega_{p,q}$ $\forall s \in (-\varepsilon, \varepsilon)$

Its variational vector field $V := \frac{\partial f}{\partial s}\big|_{s=0}$ satisfies $V(0)=0, V(1)=0$.

Conversely, if V is a VF along $c \in \Omega_{p,q}$ with $V(0)=0$, $V(1)=0$, there exists a variation of f with variational vf V: Define $f(s,t) = \exp_{c(t)}(sV(t))$ and let $\varepsilon > 0$ be so small that $f(s,t)$ is def $\forall s \in (-\varepsilon, \varepsilon)$



For a variation f of $c \in \Omega_{p,q}$, let $E(s) := E(f(s,.)) = \int_0^1 \| \frac{\partial f}{\partial t}(s,t) \|^2 dt$

<u>Thm 1.7.2</u> Let f be a proper variation of $c \in \Omega_{p,q}$ with variational vf V. Then,

$$\dot{E}(0) = -2\int_0^1 \langle V(t), \tfrac{\nabla}{dt}\dot{c}(t) \rangle dt \qquad (1.7.1)$$

In particular, $c \in \Omega_{p,q}$ is a geodesic $\Leftrightarrow$ $\dot{E}(0)$ for all proper variations of c.

<u>Pf</u>

$$\dot{E}(s) = \frac{d}{ds}\int_0^1 \langle \tfrac{\partial f}{\partial t}, \tfrac{\partial f}{\partial t} \rangle dt$$

$$= 2\int_0^1 \langle \tfrac{\nabla}{ds}\tfrac{\partial f}{\partial t}, \tfrac{\partial f}{\partial t} \rangle dt \overset{\text{Lem}}{\underset{1.2.5}{=}} 2\int_0^1 \langle \tfrac{\nabla}{dt}\tfrac{\partial f}{\partial s}, \tfrac{\partial f}{\partial t} \rangle dt$$

$$= 2\int_0^1 \tfrac{d}{dt}\langle \tfrac{\partial f}{\partial s}, \tfrac{\partial f}{\partial t} \rangle - \langle \tfrac{\partial f}{\partial s}, \tfrac{\nabla}{dt}\tfrac{\partial f}{\partial t} \rangle dt$$

$$= 2\langle \tfrac{\partial f}{\partial s}, \tfrac{\partial f}{\partial t} \rangle \Big|_{t=0}^{t=1} - \int \langle \tfrac{\partial f}{\partial s}, \tfrac{\nabla}{dt}\tfrac{\partial f}{\partial t} \rangle dt$$

Inserting $s=0$, variation proper $\Rightarrow$ (1.7.1)

c geod. $\Rightarrow \frac{\nabla}{dt}\dot{c} = 0 \Rightarrow \dot{E}(0) = 0$ for all proper variations

If c is not a geod. $\frac{\nabla}{dt}\dot{c}(t_0) \neq 0$ for some $t_0 \in (0,1)$. Let $g \in C_c^\infty((0,1))$ with $g \geq 0$ and $g(t_0) = 1$. For a proper variation of c with $V = g \cdot \frac{\nabla}{dt}\dot{c}$, we then have

$$\dot{E}(0) = -2\int_0^1 \underbrace{g \cdot \| \tfrac{\nabla}{dt}\dot{c} \|^2}_{\geq 0 \text{ and } > 0 \text{ in } t_0} dt < 0$$

$\square$

<u>Thm 1.7.3</u> Let $\gamma \in \Omega_{p,q}$ be a geod., f a proper variation of c with variational vf V. Then,

$$\ddot{E}(0) = -2\int_0^1 \langle \tfrac{\nabla^2}{dt^2}V + R(V,\dot{\gamma})\dot{\gamma}, V \rangle dt$$

<u>Pf</u>: $(1.7.1) \Rightarrow \dot{E}(s) = -2\int_0^1 \langle \tfrac{\partial f}{\partial s}, \tfrac{\nabla}{dt}\tfrac{\partial f}{\partial t} \rangle dt$

$$\Rightarrow \ddot{E}(0) = -2\int_0^1 \Big[\langle \tfrac{\nabla}{ds}\tfrac{\partial f}{\partial s}, \underbrace{\tfrac{\nabla}{dt}\tfrac{\partial f}{\partial t}}_{\substack{=0 \text{ at} \\ s=0}} \rangle + \langle \tfrac{\partial f}{\partial s}, \underbrace{\tfrac{\nabla}{ds}\tfrac{\nabla}{dt}\tfrac{\partial f}{\partial t}}_{\tfrac{\nabla^2}{dt^2}\tfrac{\partial f}{\partial s} + R(\tfrac{\partial f}{\partial s}, \tfrac{\partial f}{\partial t})\tfrac{\partial f}{\partial t}} \rangle \Big] \Big|_{s=0} dt \quad \left(\text{Pf of Thm 1.4.2}\right)$$

Thm 1.7.4 (Bonnet-Myers) Let M^n be a connected complete Riem. mfld satisfying

$$\mathrm{Ric}(v,v) \geq (n-1)\kappa\langle v,v\rangle \quad \forall v \in TM \text{ and some constant } \kappa > 0 \qquad (1.7.2)$$

Then, M is compact and $\operatorname{diam}(M) := \sup_{p,q\in M} d(p,q) \leq \frac{\pi}{\sqrt{\kappa}}$

Furthermore, $\pi_1(M)$ is finite

Pf $d(p,q) \leq \frac{\pi}{\sqrt{\kappa}} \ \forall p,q \in M$:

Assume that $d(p,q) > \frac{\pi}{\sqrt{\kappa}}$. $\overset{M}{\underset{\text{complete}}{\Longrightarrow}} \exists$ geod $\gamma \in \Omega_{p,q}$ with $L(\gamma) = \ell = d(p,q) > \frac{\pi}{\sqrt{\kappa}}$

Let $\{E_i\}_{1\leq i\leq n}$ be parallel vfds forming an ONB for each $T_{\gamma(t)}M$. Assume $E_n = \frac{\dot\gamma}{\ell}$

For $j \in \{1,\dots,n-1\}$, let $V_j(t) = \sin(\pi t) E_j(t)$

$V_j(0) = 0$, $V_j(1) = 0 \Rightarrow \exists$ proper variation f_j of γ with variational vf V_j

Consider $E_j(s) = E(f_j(s,.))$. Thm 1.7.2 $\Rightarrow \dot E_j(0) = 0$

$$\text{Thm 1.7.3} \Rightarrow \tfrac{1}{2}\ddot E_j(0) = -\int_0^1 \langle \tfrac{\nabla^2}{dt^2}V_j + R(V_j,\dot\gamma)\dot\gamma, V_j\rangle dt$$

$$= \int_0^1 \sin^2(\pi t)\langle \pi^2 E_j - R(E_j,\dot\gamma)\dot\gamma, E_j\rangle dt$$

Observe that $\sum_{j=1}^{n-1}\langle R(E_j,\dot\gamma)\dot\gamma, E_j\rangle = \sum_{j=1}^{n}\langle R(X_j,\dot\gamma)\dot\gamma, E_j\rangle - \underbrace{\langle R(\tfrac{\dot\gamma}{\ell},\dot\gamma)\dot\gamma, \tfrac{\dot\gamma}{\ell}\rangle}_{=0} = \mathrm{Ric}(\dot\gamma,\dot\gamma)$

$$\Rightarrow \sum_{j=1}^{n-1}\tfrac{1}{2}\ddot E_j(0) = \int_0^1 \sin^2(\pi t)[(n-1)\pi^2 - \mathrm{Ric}(\dot\gamma,\dot\gamma)]\,dt$$

$$\leq \int_0^1 \sin^2(\pi t)(n-1)\big(\underbrace{\pi^2 - \kappa\langle\dot\gamma,\dot\gamma\rangle}_{\pi^2 - \frac{\kappa}{\ell^2} < 0 \text{ by assumption } \ell \geq \frac{\pi}{\sqrt{\kappa}}}\big)dt < 0$$

$\exists j_0 \in \{1,\dots,n-1\}$ s.t. $\ddot E_{j_0}(0) < 0 \Rightarrow E_{j_0}(f(s,.)) < E(\gamma)$ for $s \neq 0$ small

$\lightning$ contradicts Lem 1.7.1 and γ minimizing $\rightarrow \operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\kappa}}$

$\Rightarrow M$ closed & bded $\Rightarrow M$ cpt by Hopf-Rinow thm

$\pi_1(M)$ finite: Let $\pi: \tilde M \to M$ be the universal cover, $\tilde M$ equipped with the metric $\pi^* g$

π local isometry $\Rightarrow \tilde M$ satisfies (1.7.2) $\Rightarrow \tilde M$ is cpt. $\Rightarrow$ For $p \in M$, the discrete subset $\pi^{-1}(p) \subset \tilde M$ must be finite $\Rightarrow \pi_1(M)$ is finite.

Rem 1.7.5 The sphere of radius r satisfies $Ric(v,v) = \frac{n-1}{r^2}\langle v,v\rangle$ and $diam(M) = r^2\pi$ $\Rightarrow$ equality in Thm 1.7.4 with $K = \frac{1}{r^2}$.

One can actually show that under the assumptions of Thm 1.7.4, $diam(M) = \frac{\pi}{\sqrt{K}}$ $\Leftrightarrow$ M^n is isometric to S^n.

Rem 1.7.6 If M is cpt, (1.7.2) is equivalent to $Ric(v,v) > 0$ $\forall v \in TM$

$\Rightarrow$ T^n does not admit a metric with $Ric > 0$ as $\pi_1(T^n)$ is infinite