Gauss–Bonnet theorem

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October 2023

1 Introduction

The Gauss-Bonnet theorem is a profound result in differential geometry that establishes a remarkable connection between the geometry of a surface and its topology. Named after Carl Friedrich Gauss and Pierre Ossian Bonnet, this theorem encapsulates the intrinsic curvature of a two-dimensional manifold.

At its core, the theorem relates the integral of the Gaussian curvature over a surface to topological invariants. For a closed, orientable surface (think of a compact, boundary-free shape like a sphere or a torus), the Gauss-Bonnet theorem states that the integral of the Gaussian curvature is equal to 2π times the Euler characteristic of the surface.

Mathematically, the theorem can be expressed as:

$$\int_S K \, dA = 2\pi\chi(M) \tag{1}$$

Here, S represents the surface, K is the Gaussian curvature, dA is the area element, and $\chi(M)$ denotes the Euler characteristic of the underlying manifold M. The Euler characteristic is a topological invariant that captures the essence of the surface's "holes" or handles.

In essence, the Gauss-Bonnet theorem beautifully connects local geometry (Gaussian curvature) with global topology (Euler characteristic). This bridge between differential and topological properties has profound implications in various fields, including physics and mathematics. It's a testament to the deep interplay between geometry and topology, revealing the intrinsic beauty and unity in mathematical structures. In this note, we're handing you the mathematical tools you need to dive into the proof of (1), **fully based on** [1].

Rotation Index and Curved Polygon

Consider the function $\gamma : [a, b] \to \mathbb{R}^2$ representing an admissible curve within the plane. We call γ to be a *simple closed curve* if it adheres to the condition $\gamma(a) = \gamma(b)$ while maintaining injectivity across the interval $[a, b]$. The *unit tangent vector field* along γ is denoted as T and is defined on each smooth segment as follows:

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

Given that each tangent space to $\mathbb{R}^2$ naturally corresponds to $\mathbb{R}^2$ itself, we conceptualize T as a mapping into $\mathbb{R}^2$. Notably, as a consequence of its unit length, T takes values within the unit circle S^1.

In instances where γ is a smooth (or at least continuously differentiable) curve, we introduce a *tangent angle function for* γ as a continuous function $\theta : [a, b] \to \mathbb{R}$ such that $T(t) = (\cos\theta(t), \sin\theta(t))$ for all $t \in [a, b]$. The existence of such a function is derived from the theory of covering spaces, where the map $q : \mathbb{R} \to S^1$ defined by $q(s) = (\cos s, \sin s)$ serves as a smooth covering map. The path-lifting property of covering maps ensures the existence of a continuous function $\theta : [a, b] \to \mathbb{R}$ satisfying $q(\theta(t)) = T(t)$. The unique lifting property establishes the uniqueness of a lift once its value at any single point is determined, with any two lifts differing by a constant integral multiple of 2π.

In the scenario where γ is a continuously differentiable simple closed curve with $\gamma'(a) = \gamma'(b)$, we observe that $(\cos\theta(a), \sin\theta(a)) = (\cos\theta(b), \sin\theta(b))$, resulting in $\theta(b) - \theta(a)$ being an integral multiple of 2π. For such cases, we introduce the *rotation index of* γ as the following integer:

$$\rho(\gamma) = \frac{1}{2\pi} \left(\theta(b) - \theta(a)\right).$$

Here, θ represents any tangent angle function for γ. Notably, the rotation index remains independent of the chosen tangent angle function, as $\theta(a)$ and $\theta(b)$ undergo changes by the addition of the same constant for any alternative θ.

Rotation Index of Piece-wise Regular Closed Curves

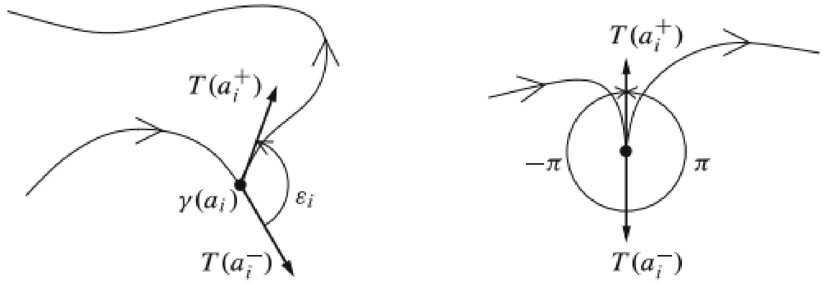
Assume $\gamma : [a, b] \to \mathbb{R}^2$ is an admissible simple closed curve, and $(a_0, \ldots, a_k)$ be an admissible partition of $[a, b]$. The possible singular (non-smooth) points of $\gamma(a_i)$ are called the *vertices of* γ and each piece $\gamma|_{[a_{i-1}, a_i]}$ is called an *edge* or a *side*. We can investigate three scenarios (recall that at any vertex a_i on the curve there are left-hand and right-hand velocity vectors $\gamma'(a_i^-)$ and $\gamma'(a_i^+)$ respectively. These are tangent, with respect to the orientation of γ, to the "incoming and outcoming edges" respectively. See Figure 1 for details.):

1. If $T(a_i^-) \neq \pm T(a_i^+)$, then $\gamma(a_i)$ is an *ordinary vertex.*
2. If $T(a_i^-) = T(a_i^+)$, then $\gamma(a_i)$ is a *flat vertex.*
3. If $T(a_i^-) = -T(a_i^+)$, then $\gamma(a_i)$ is a *cusp vertex.*

Now, we can define the *exterior angle at* $\gamma(a_i)$ to be the oriented measure ϵ_i of the angle from $T(a_i^-)$ to $T(a_i^+)$, chosen to be in the interval $(-\pi, \pi)$, with a positive sign if $\left(T(a_i^-), T(a_i^+)\right)$ is an oriented basis for $\mathbb{R}^2$, and a negative sign otherwise; see Figure 1. Throughout this note, we exclude the possibility of a cusp vertex in γ since the definition of the exterior angle is not well-defined in such cases.

Definition 1.1. *A curved polygon in the plane is an admissible simple closed curve without cusp vertices, whose image is the boundary of a precompact open set $\Omega \subset \mathbb{R}^2$. The set Ω is called the interior of γ.*

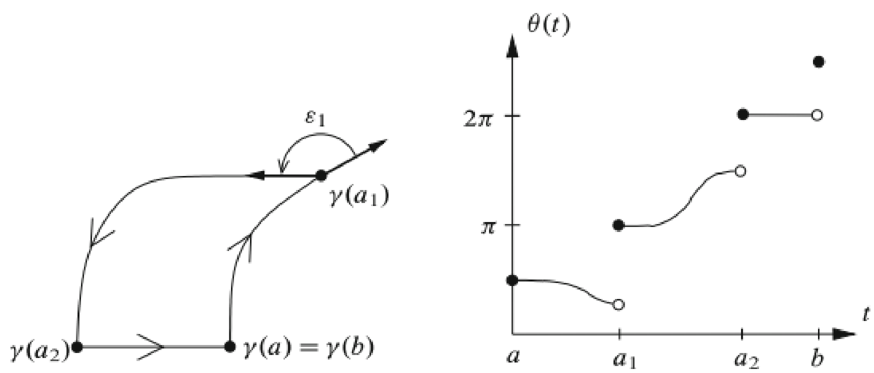
Let $\gamma : [a, b] \to \mathbb{R}^2$ represent a curved polygon. If γ is parameterized in such a way that at its smooth points, the tangent vector $\gamma'(t)$ aligns positively with the induced orientation on $\partial\Omega$ as per Stokes's theorem, we designate γ as positively oriented. In simpler terms, this implies that



An exterior angle   A cusp vertex

Figure 1: [1] Fig. 9.2 & Fig. 9.3. The above are depictions of an exterior angle and a cusp vertex. Notice how the vectors $T(a_i^+)$ and $T(a_i^-)$ are tangent to different edges of the curve γ.

γ is parametrized in the counterclockwise direction, or, in an intuitive sense, that Ω consistently resides to the left of γ. Before stating rotation index for a curved polygon $\gamma : [a, b] \to \mathbb{R}^2$, we need the notion of tangent angle function for γ. This can be done by defining $\theta(a_i) = \lim_{t \to a_i^-} \theta(t) + \epsilon_i$ and $\theta(b) = \lim_{t \to b^-} \theta(t) + \epsilon_k$, where ϵ_i is the exterior angle at $\gamma(a_i)$. For this curve, we define the rotation index of γ to be $\rho(\gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a))$; see (2).

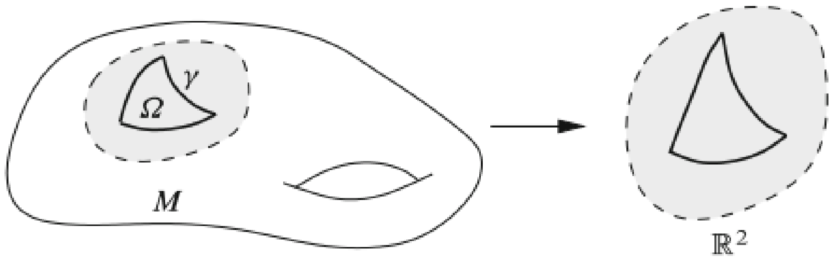


Tangent angle at a vertex   Tangent angle function

Figure 2: [1] Fig. 9.5 & Fig. 9.6

Theorem 1.1 ([1], Theorem 9.1)**.** *The rotation index of a positively oriented curved polygon in the plane is* $+1$.

The Gauss-Bonnet Formula



The concept of curved polygon and rotation index can be viewed on an oriented Riemannian 2-manifold (M, g) instead of the plane.

Curved polygon on M: Take an admissible simple closed curve $\gamma : [a, b] \to M$ such that the image of γ is the boundary of a precompact open set $\Omega \subseteq M$, and there is an oriented smooth coordinate disc $\overline{\Omega}$ such that the image of γ is a curved polygon in the corresponding chart; see Section 1.

Exterior angle of γ at $\gamma(a_i)$: Let $T(t) = \gamma'(t)/|\gamma'(t)|_g$ then the oriented measure ϵ_i is defined by the angle from $T(a_i^-)$ to $T(a_i^+)$ with respect to the g-inner product and the given orientation of M i.e.,

$$\epsilon_i = \frac{dV_g(T(a_i^-), T(a_i^+))}{|dV_g(T(a_i^-), T(a_i^+))|} \arccos\langle T(a_i^-), T(a_i^+)\rangle_g. \tag{2}$$

Now, we seek a curved polygon rotation index theorem tailored for M. Imagine $\gamma\colon [a, b] \to M$ as a curved polygon with Ω as its interior. Let (U, ϕ) be a smoothly oriented chart containing $\overline{\Omega}$. Using the coordinate map ϕ to project γ, Ω, and g onto the plane, we can assume g is a metric on an open subset $\hat{U} \subset \mathbb{R}^2$, and γ is a curved polygon in $\hat{U}$. Define an oriented orthonormal frame (E_1, E_2) for g via the Gram–Schmidt algorithm on (∂_x, ∂_y), ensuring E_1 is a positive scalar multiple of ∂_x throughout $\hat{U}$.

We introduce a tangent angle function for γ as a piecewise continuous function $\theta\colon [a, b] \to \mathbb{R}$ satisfying

$$T(t) = \cos\theta(t) E_1|_{\gamma(t)} + \sin\theta(t) E_2|_{\gamma(t)}$$

where γ' is continuous, and it is continuous from the right at vertices. The existence of such a function follows as in the planar case, utilizing the fact that

$$T(t) = u_1(t) E_1|_{\gamma(t)} + u_2(t) E_2|_{\gamma(t)}$$

for piecewise continuous functions $u_1, u_2\colon [a, b] \to \mathbb{R}$ viewed as coordinate functions of a map $(u_1, u_2)\colon [a, b] \to S^1$ since T has unit length.

The rotation index of γ is $\rho(\gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a))$. Despite the role of the specific frame (E_1, E_2) in the definition, it's not immediately evident that the rotation index has any coordinate-independent meaning. However, an easy consequence of the rotation index theorem shows that it is independent of coordinate choice.

Lemma 1.1 (Lemma 9.2, [1])**.** *For an oriented Riemannian 2-manifold M, the rotation index of every positively oriented curved polygon in M is $+1$.*

From this point onward, we assume for convenience that our curved polygon γ is given a unit-speed parametrization, so the unit tangent vector field $T(t)$ is equal to $\gamma'(t)$. There is a unique unit normal vector field N along the smooth portions of γ such that $(\gamma'(t), N(t))$ is an oriented orthonormal basis for $T_{\gamma(t)}M$ for each t. If γ is positively oriented as the boundary of Ω, this is equivalent to N being the inward-pointing normal to $\partial\Omega$. We define the signed curvature of Ω at smooth points of γ by

$$\kappa_N(t) = \langle D_t\gamma'(t), N(t)\rangle_g.$$

By differentiating $|\gamma'(t)|^2 = 1$, we see that $D_t\gamma'(t)$ is orthogonal to $\gamma'(t)$, and therefore we can write $D_t\gamma'(t) = \kappa_N(t)N(t)$, and the (unsigned) geodesic curvature of γ is $\kappa(t) = |\kappa_N(t)|$. The sign of $\kappa_N(t)$ is positive if γ is curving toward Ω, and negative if it is curving away.

Theorem 1.2 (The Gauss–Bonnet Formula)**.** *Let (M, g) be an oriented Riemannian 2-manifold. Suppose γ is a positively oriented curved polygon in M, and Ω is its interior. Then*

$$\int_\Omega K\,dA + \int_\gamma \kappa_N ds + \sum_{i=1}^{k} \epsilon_i = 2\pi, \tag{3}$$

where K is the Gaussian curvature of g, dA is its Riemannian volume form, and $\theta_1, \ldots, \theta_k$ are the exterior angles of γ. The second integral is taken with respect to arc length.

Proof. Let $(a_0, \ldots, a_k)$ be an admissible partition of $[a, b]$, and let (x, y) be oriented smooth coordinates on an open set U containing $\overline{\Omega}$. Let $\theta : [a, b] \to \mathbb{R}$ be a tangent angle function for γ. Using the rotation index theorem one can write

$$2\pi = \theta(b) - \theta(a) = \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \theta'(t)dt. \tag{4}$$

Let (E_1, E_2) be the oriented g-orthonormal frame as before. At smooth points of γ the following formula holds:

$$\gamma'(t) = \cos\theta(t)E_1|_{\gamma(t)} + \sin\theta(t)E_2|_{\gamma(t)} \tag{5}$$

$$N(t) = -\sin\theta(t)E_1|_{\gamma(t)} + \cos\theta(t)E_2|_{\gamma(t)} \tag{6}$$

By differentiating γ' we get

$$D_t\gamma' = -(\sin\theta)\theta' E_1 + (\cos\theta)\nabla_{\gamma'}E_1 + (\cos\theta)\theta' E_2 + (\sin\theta)\nabla_{\gamma'}E_2 = \theta' N + (\cos\theta)\nabla_{\gamma'}E_1 + (\sin\theta)\nabla_{\gamma'}E_2 \tag{7}$$

Now, we are interested in covariant derivatives of E_1 and E_2. Since (E_1, E_2) is an orthonormal frame, for every vector v, thus

$$\begin{aligned} 0 &= \nabla_v |E_1|^2 = 2\langle \nabla_v E_1, E_1\rangle, \\ 0 &= \nabla_v |E_2|^2 = 2\langle \nabla_v E_2, E_2\rangle, \\ 0 &= \nabla_v (E_1, E_2) = \langle \nabla_v E_1, E_2\rangle + \langle E_1, \nabla_v E_2\rangle. \end{aligned}$$

The first two equations show that $\nabla_v E_1$ is a multiple of E_2 and $\nabla_v E_2$ is a multiple of E_1. Define a 1-form ω by

$$\omega(v) = \langle E_1, \nabla_v E_2\rangle = -\langle \nabla_v E_2, E_1\rangle$$

It follows that the covariant derivatives of the basis vectors are given by

$$\nabla_v E_1 = -\omega(v)E_2; \tag{8}$$

$$\nabla_v E_2 = \omega(v)E_1 \tag{9}$$

Therefore, we can compute the signed curvature of γ as follows:

$$\begin{aligned} \kappa_N &= \langle D_t\gamma', N\rangle \\ &= \langle \theta' N, N\rangle + \cos\theta\langle \nabla_{\gamma'} E_1, N\rangle + \sin\theta\langle \nabla_{\gamma'} E_2, N\rangle \\ &= \theta' - \cos\theta\langle \omega(\gamma')E_2, N\rangle + \sin\theta\langle \omega(\gamma')E_1, N\rangle \\ &= \theta' - \cos^2\theta\omega(\gamma') - \sin^2\theta\omega(\gamma') \\ &= \theta' - \omega(\gamma'). \end{aligned}$$

Therefore, (4) becomes

$$\begin{aligned} 2\pi &= \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \kappa_N(t)dt + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \omega(\gamma'(t))dt \\ &= \sum_{i=1}^{k} \epsilon_i + \int_\gamma \kappa_N ds + \int_\gamma \omega. \end{aligned}$$

So, we only need to prove

$$\int_\gamma \omega = \int_\Omega K dA. \tag{10}$$

Because Ω is a smooth manifold with corners, we can apply Stokes's theorem and conclude that the left-hand side of (10) is equal to $\int_\Omega d\omega$. The last step of the proof is to show that $d\omega = KdA$. Since (E_1, E_2) is an oriented orthonormal frame, we have that that $dA(E_1, E_2) = 1$ (see [1], Prop 2.41). Therefore, we have

$$\begin{aligned} KdA(E_1, E_2) &= K \\ &= R(E_1, E_2, E_2, E_1) = \langle \nabla_{E_1}\nabla_{E_2}E_2 - \nabla_{E_2}\nabla_{E_1}E_2 - \nabla_{[E_1,E_2]}E_2, E_1\rangle \\ &= \langle \nabla_{E_1}(\omega(E_2)E_1) - \nabla_{E_2}(\omega(E_1)E_1) - \omega([E_1, E_2])E_1, E_1\rangle \\ &= \langle E_1(\omega(E_2))E_1 + \omega(E_2)\nabla_{E_1}E_1 - E_2(\omega(E_1))E_1 - \omega(E_1)\nabla_{E_2}E_1 - \omega([E_1, E_2])E_1, E_1\rangle \\ &= E_1(\omega(E_2)) - E_2(\omega(E_1)) - \omega([E_1, E_2]) \\ &= d\omega(E_1, E_2). \end{aligned}$$

$\square$

The global Gauss-Bonnet

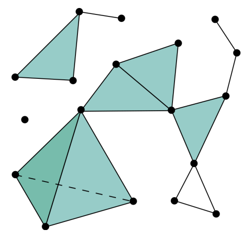


Figure 3: An example of a simplicial complex

Recall first, that a simplicial complex K is a set of simplices such that for every $\sigma \in K$

- For every face τ of σ, we have $\tau \in K$
- If $\sigma_1, \sigma_2 \in K$ and $\tau = \sigma_1 \cap \sigma_2$ is non-empty, then τ is a face of both σ_1 and σ_2.

Let X be any (topological) manifold. A triangulation of X is a pair $(\tilde{X}, h)$, where $\tilde{X}$ is the realization of some simplicial complex and $h \colon \tilde{X} \to X$ is a homeomorphism. Similarly, we may define smooth triangulations; for general smooth manifolds X, a smooth triangulation is a triangulation s.t $h|\sigma \colon \sigma \to X$ is a smooth embedding. Let M be a smooth, compact Riemannian 2-manifold; the definition of smooth triangulation will be more restrictive.

Definition 1.2.
- *A curved triangle in M is a curved polygon with exactly three vertices and three edges.*
- *A smooth triangulation of M is a finite collection of curved triangles with disjoint interiors, such that the union of the triangles with their interiors is M, and the (if non-empty) intersection of any pair of triangles is either a vertex in each or a single edge of each.*

In general, a manifold M may have many different triangulations (which are of course same up to homeomorphism).

Remark. The definition of a smooth triangulation is precisely that of a triangulation, where the simplicial complex is built out of curved triangles. One should think of a smooth triangulation as approximating a manifold via a smooth simplicial complex.

The requirement to be smoothly triangulable is not restrictive; any smooth, compact surface has a smooth triangulation. In fact, every compact topological 2-manifold has a (not necessarily smooth) triangulation and any smooth n-manifold has a smooth triangulation (this is a theorem of Whitehead).

In order to pass from the Gauss-Bonnet formula, which is a local statement, to the Gauss-Bonnet theorem, which is a global statement, we will use smooth triangulations on M. Suppose in the sequel that M is triangulated.

Definition 1.3. *The Euler characteristic χ of M with respect to a given triangulation is*

$$\chi(M) = V - E + F,$$

where V, E, F are respectively the numbers of vertices, edges and faces in the triangulation.

The Euler characteristic is invariant under homeomorphism (even homotopy equivalence), so thus in particular under diffeomorphisms and isometries. Thus, the choice of triangulation does not matter.

Theorem 1.3 (Gauss-Bonnet, [1] Thm. 9.7)**.** *If (M, g) is a smoothly triangulated compact Riemannian 2-manifold, then*

$$\int_M KdA = 2\pi\chi(M),$$

where K is the Gaussian curvature of M and dA is its Riemannian density.

If M is orientable, the Riemannian density dA is given as $|dV_g|$, where dV_g is the Riemannian volume form. The proof will be split into two cases, based on whether M is orientable or not. Note, that we may assume M is connected; if not, we integrate over each component and take the sum of the integrals.

Proof when M is orientable. Suppose M is orientable. Choose an orientation and interpret the integral $\int_M KdA$ as integration with respect to the Riemannian volume form (since M is orientable, this is equal to integrating w.r.t the density). Let $\{\Omega_i \mid i = 1, \ldots, F\}$ denote the faces of the triangulation, and for each i, let $\{\gamma_{ij} \mid j = 1, 2, 3\}$ denote the edges of Ω_i and $\{\theta_{ij} \mid j = 1, 2, 3\}$ its interior angles. Each exterior angle is $\pi - \theta_{ij}$, so applying the Gauss-Bonnet formula to each triangle and summing over i we have

$$\sum_{i=1}^{F} \int_{\Omega_i} KdA + \sum_{i=1}^{F}\sum_{j=1}^{3} \int_{\gamma_{ij}} \kappa_N ds + \sum_{i=1}^{F}\sum_{j=1}^{3} (\pi - \theta_{ij}) = \sum_{i=1}^{F} 2\pi. \tag{11}$$

Suppose we look at a single triangle Ω_i and its edges $\gamma_{i1}, \gamma_{i2}, \gamma_{i3}$. Then to each edge there will correspond another face of the triangulation, which will necessarily have opposite orientation. Thus all the edge integrals in (11) cancel out. Since the Ω_i cover M, we simplify (11) to get

$$\int_M KdA + 3\pi F - \sum_{i=1}^{F}\sum_{j=1}^{3} \theta_{ij} = 2\pi F. \tag{12}$$

Note, that similarly each interior angle θ_{ij} appears exactly once. Furthermore, the interior angles at a vertex have to add up to 2π, so we may rearrange the sum of the angles in (12) to get

$$\int_M KdA = 2\pi V - \pi F. \tag{13}$$

Since each edge appears in exactly two triangles and each triangle has exactly three edges, the total number of edges counted with multiplicity is $2E = 3F$; we count each edge once for each triangle it appears in. Thus $F = 2E - 2F$, so (13) becomes

$$\int_M KdA = 2\pi V - 2\pi F + 2\pi E = 2\pi\chi(M).$$

□

For the non-orientable case, we need more tools.

Proposition 1 ([2] Thm. 15.41, [1] B.18)**.** *If M is a connected, non-orientable smooth manifold, then there exists an oriented smooth manifold $\hat{M}$ and a two-sheeted smooth covering map $\hat{\pi}\colon \hat{M} \to M$.*

This covering is often called the orientation double cover; the existence of the orientation double cover in particular means we can always pass to an orientable manifold.

One can show given a finite covering $\hat{\pi}\colon \hat{M} \to M$, that compactness of M implies compactness of $\hat{M}$ (in fact, this can be refined to say that M is compact and $\hat{M} \to M$ is a finite sheeted covering if and only if $\hat{M}$ is compact).

Proof when M is non-orientable. Choose an oriented smooth manifold $\hat{M}$ covering M as above, so that $\hat{M}$ is compact. We give $\hat{M}$ the pullback metric $\hat{g} = \hat{\pi}^* g$. Then the Riemann density of $\hat{M}$ is $\widehat{dA} = \hat{\pi}^* dA$ and the curvature is $\hat{K} = \hat{\pi}^* K$. Thus $\hat{\pi}^*(KdA) = \hat{K}\widehat{dA}$. One can show that for a k-sheeted Riemannian covering $\hat{M} \to M$ of connected compact Riemannian manifolds there is an equality

$$Vol(\hat{M}) = k \cdot Vol(M).$$

Thus, we get

$$\int_{\hat{M}} \hat{K}\widehat{dA} = 2\int_M KdA,$$

as our chosen covering is 2-sheeted. All that is left is to compare the Euler characteristics of $\hat{M}$ and M. To do this, we show that the triangulation of M lifts to a triangulation on $\hat{M}$. Let γ be any curved triangle in M with interior Ω. By definition, there exists a smooth chart (U, ϕ) whose domain contains $\bar{\Omega}$ and whose image is a disk $D \subset \mathbb{R}^2$, where $\phi(\bar{\Omega}) = \bar{\Omega}_0$ and $\bar{\Omega}_0$ is the interior of a curved triangle γ_0 in $\mathbb{R}^2$. Now ϕ^{-1} is an embedding of D into M, which restricts to a diffeomorphism $F\colon \bar{\Omega}_0 \to \bar{\Omega}$. Covering space theory tells us the following: because D is simply connected, ϕ^{-1} has a lift to $\hat{M}$ which is smooth, since $\hat{\pi}$ is a local diffeomorphism. Because the covering is 2-sheeted, there are exactly two such lifts F_1 and F_2. Each lift is injective, since $\hat{\pi} \circ F_i = F$ and their images are disjoint, since if they were to agree at a point, the lifts would be the same. From this one verifies that the triangulation of $\hat{M}$ given by lifting curved triangles of M using the maps F_i gives a triangulation which has exactly two times the number of vertices, edges and faces. Thus $\chi(\hat{M}) = 2\chi(M)$. Thus, applying the orientable Gauss-Bonnet theorem for the manifold $\hat{M}$, we get

$$\int_M KdA = \frac{1}{2}\int_{\hat{M}} \hat{K}\widehat{dA} = \frac{1}{2} \cdot 2\pi\chi(\hat{M}) = \pi\chi(\hat{M}) = 2\pi\chi(M),$$

which finishes the proof. □

The Gauss-Bonnet theorem has significant implications not only on the possible Gaussian curvatures of compact surfaces but also on the topological properties of the given manifold. Recall that every compact, connected orientable 2-manifold M is homeomorphic to a sphere or a connected sum of n tori, and every non-orientable such manifold is homeomorphic to a connected sum of n copies of $\mathbb{R}P^2$. The number n is called the genus of M; one can show that $\chi(M) = 2 - 2n$ for an orientable surface and $\chi(M) = 2 - n$ for a non-orientable surface. We obtain the following corollary.

Corollary 1. *Let (M, g) be a compact Riemannian 2-manifold and let K be its Gaussian curvature.*

1. *If M is homeomorphic to the sphere or the projective plane, then $K > 0$ for some point on M.*

2. *If M is homeomorphic to the torus or the Klein bottle, then $K = 0$, or K takes on both positive and negative values.*

3. *If M is any other compact surface, then $K < 0$ for some point on M.*

Proof. The proof is immediate from the Gauss-Bonnet theorem and knowledge of the Euler characteristics of the given surfaces. □

The corollary has a deep converse: if K is any smooth function on a compact 2-manifold M with the sign conventions of the above corollary, then there is a Riemannian metric g on M, such that K is the Gaussian curvature of M.

Exercises

Exercise 1

Let (M, g) be a compact Riemannian 2-manifold and K its Gaussian curvature.

1. If $K > 0$ everywhere on M, then the universal covering $\tilde{M}$ of M is (homeomorphic to) S^2 (the 2-sphere), and $\pi_1(M)$ is either trivial or (isomorphic to) $\mathbb{Z}/2$.

2. If $K \leq 0$ everywhere on M, then $\tilde{M}$ is (homeomorphic to) $\mathbb{R}^2$ and $\pi_1(M)$ is infinite.

Hint: use the classification theorem of compact surfaces and knowledge of the Euler characteristics of compact surfaces. In 2), use the fact that any fiber of a covering $\tilde{M} \to M$ has the cardinality of $\pi_1(M)$.

Exercise 2

Let $M \subset \mathbb{R}^3$ be a compact, embedded, 2-dimensional Riemannian submanifold. Show that M cannot have $K \leq 0$ everywhere. Hint: look at a point $p \in M$ where the distance from the origin is maximized; study the principal curvatures at p.

Exercise 3

Let $M \subset \mathbb{R}^3$ be a compact, connected, regular orientable 2-dimensional Riemannian submanifold, which is not homeomorphic to a sphere. Show that M has points where the Gaussian curvature is negative, positive and zero. Hint: use Exercise 2 and the Gauss-Bonnet theorem.

References

[1] M John. Lee, riemannian manifolds: an introduction to curvature. *Graduate Texts in Mathematics*, 176, 1997.

[2] John M. Lee. *Introduction to smooth manifolds*. Springer, 2012.