Spaces of constant curvature

Benedetta Andina (SU)

October 13, 2023

today:

- Cartan's Theorem, and the specific case of constant curvature manifolds;
- Main theorem:

  Theorem. *Any complete Riemannian manifold with constant curvature has its universal covering space, with the covering metric, isometric to* $\mathbb{H}^n$*,* $\mathbb{R}^n$ *or* $\mathbb{S}^n$*.*

- Some group theory and covering spaces.

Remark 0.1. If you dilate the metric by c, then the sectional curvature gets multiplied by $\frac{1}{c}$.

Thus we are just going to assume that the possible constant curvatures are $-1, 0, 1$.

1 Theorem of Cartan

Let M and $\tilde{M}$ be two n-dimensional Riemannian manifolds and $p \in M$, $\tilde{p} \in \tilde{M}$ two points. Let $\iota : T_pM \to T_{\tilde{p}}\tilde{M}$ be a linear isometry and V be a neighborhood of p such that $\exp_{\tilde{p}}$ is defined on $\iota(\exp_p^{-1}(V))$. Define $f : V \to \tilde{M}$ as

$$f := \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1}.$$

In order to define a second map that we are going to use, note that for every $q \in V$ there exists a unique normalized geodesic

$$\begin{aligned} \gamma : [0, l] &\to M \\ 0 &\mapsto p \\ l &\mapsto q \end{aligned} .$$

We also need to define for every q the parallel transport along γ $P_l : T_pM \to T_qM$ from $p = \gamma(0)$ to $q = \gamma(l)$ and $\tilde{P}_l$ the parallel transport along $\tilde{\gamma} : [0, l] \to \tilde{M}$

(the unique normalized geodesic with $\tilde{\gamma}(0) = \tilde{p} = f(p), \tilde{\gamma}(l) = f(q), \tilde{\gamma}'(0) = \iota(\gamma'(0))$). We then can define the map

$$\phi_l = \tilde{P}_l \circ \iota \circ P_l^{-1} : T_q M \to T_{f(q)} \tilde{M}$$

which is a linear isometry (it is the composition of linear isometries). Finally, let R be the curvature of M and $\tilde{R}$ the curvature of $\tilde{M}$.

Theorem 1.1 (Cartan)**.** *With the notation above, if for every $q \in V$, $x, y, u, v \in T_q M$ we have*

$$\langle R(x, y)u, v\rangle = \langle \tilde{R}(\phi_l(x), \phi_l(y))\phi_l(u), \phi_l(v)\rangle,$$

then $f : V \to f(V)$ is a local isometry and $df_p = \iota$.

Proof. Let v be an element of $T_q M$ and J the Jacobi field along γ with $J(0) = 0$ and $J(l) = v$. Let $e_1, \ldots, e_{n-1}, \gamma'(0)(:= e_n)$ an orthonormal basis of $T_p M$ and $e_i(t) = P_t(e_i)$ for every i (this is still an orthonormal basis, of $T_{\gamma(t)} M$). Then we can write J as

$$J(t) = \sum_i y_i(t)e_i(t).$$

Since this satisfies the Jacobi equation

$$\frac{D^2}{dt^2} \sum_i y_i(t)e_i(t) + R\left(\sum_i y_i(t)e_i(t), e_n(t)\right) e_n(t) = 0,$$

it must satisfy

$$\left\langle \frac{D^2}{dt^2} \sum_i y_i(t)e_i(t) + R\left(\sum_i y_i(t)e_i(t), e_n(t)\right) e_n(t), e_j(t) \right\rangle = 0$$

for every j, so we have that

$$y_j''(t) + \sum_i \langle R(e_i(t), e_n(t))e_n(t), e_j(t)\rangle y_i(t) = 0$$

for every $j = 1, \ldots, n$.

If we consider $\tilde{J}$ the vector field along $\tilde{\gamma}$ given by $\tilde{J}(t) = \phi_t(J(t))$, $t \in [0, l]$ and $\tilde{e}_i(t) = \phi_t(e_i(t))$, by linearity of ϕ_t we get

$$\tilde{J}(t) = \phi_t\left(\sum_i y_i(t)e_i(t)\right) = \sum_i y_i(t)\phi_t(e_i(t)) = \sum_i y_i(t)\tilde{e}_i(t).$$

Since by hypothesis $\langle R(e_i(t), e_n(t))e_n(t), e_j(t)\rangle = \langle \tilde{R}(\tilde{e}_i(t), \tilde{e}_n(t))\tilde{e}_n(t), \tilde{e}_j(t)\rangle$, then also $\tilde{J}$ is a Jacobi field. Moreover,

$$\tilde{J}(0) = \phi_t(J(0)) = \phi_t(0) = 0$$

since ϕ_t is linear and

$$|\tilde{J}(t)| = |\phi_t(J(t))| = |J(t)|$$

since ϕ_t is an isometry. In particular, for $t = l$, we get that $|\tilde{J}(l)| = |J(l)| = |v|$ We claim that

$$\tilde{J}(l) = df_q(v).$$

Then, for $u, v \in T_qM$, we will get

$$\langle u, v\rangle_q = \langle \hat{J}(l), J(l)\rangle_q = \langle \tilde{\hat{J}}(l), \tilde{J}(l)\rangle_q = \langle df_q(u), df_q(v)\rangle_q,$$

for $\hat{J}$ the Jacobi field with $\hat{J}(l) = u$. Hence, f is a local isometry.

Let's now prove the claim. Using *Corollary 1.4.6*, we get

$$J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0)),$$
$$\tilde{J}(t) = (d\exp_{\tilde{p}})_{t\tilde{\gamma}'(0)}(t\tilde{J}'(0)).$$

Thus

$$l\iota(J'(0)) = \iota(lJ'(0)) = \iota\left(l\frac{(d\exp_p)^{-1}_{l\gamma'(0)}J(l)}{l}\right)$$

since ι is an isometry, and thanks to the fact that $\tilde{J}'(0) = \iota(J'(0))$,

$$\begin{aligned}\tilde{J}(l) =&(d\exp_{\tilde{p}})_{l\tilde{\gamma}'(0)}(l\tilde{J}'(0)) = (d\exp_{\tilde{p}})_{l\tilde{\gamma}'(0)}(l\iota(J'(0))) = \\ =&(d\exp_{\tilde{p}})_{l\tilde{\gamma}'(0)}(\iota(d\exp_p)^{-1}_{l\gamma'(0)}J(l)) = \\ =&(d\exp_{\tilde{p}})_{l\tilde{\gamma}'(0)} \circ \iota \circ (d\exp_p)^{-1}_{l\gamma'(0)}(J(l)) = df_q(J(l)).\end{aligned}$$

□

Remark 1.2. Note that if $\exp_p$ and $\exp_{\tilde{p}}$ are diffeomorphisms, then f is defined on M and it is an isometry.

Corollary 1.3. *Let $M, \tilde{M}$ be two n-dimensional Riemannian manifolds of constant curvature κ and $p \in M, \tilde{p} \in \tilde{M}$ two points. Suppose we have two orthonormal bases $\{e_j\}_{j=1,\ldots,n}$ of T_pM and $\{\tilde{e}_j\}_{j=1,\ldots,n}$ of $T_{\tilde{p}}\tilde{M}$. Then there exists a neighborhood V of p, a neighborhood $\tilde{V}$ of $\tilde{p}$ and a map $f : V \to \tilde{V}$ such that $df_p(e_j) = \tilde{e}_j$.*

Proof. Choose a linear isometry ι such that for every j $\iota(e_j) = \tilde{e}_j$. We want to show that the condition on the curvature of the theorem is satisfied. By *Corollary 6.2.7, Lecture notes differential geometry,* we have that the curvature is costant if and only if $\langle R(x, y)u, v\rangle = \kappa(\langle x, v\rangle\langle y, u\rangle - \langle x, u\rangle\langle y, v\rangle)$ for every $x, y, u, v \in T_qM$, so

$$\begin{aligned}\langle R(x, y)u, v\rangle =&\kappa(\langle x, v\rangle\langle y, u\rangle - \langle x, u\rangle\langle y, v\rangle) = \\ =&\kappa(\langle \phi_l(x), \phi_l(v)\rangle\langle \phi_l(y), \phi_l(u)\rangle - \langle \phi_l(x), \phi_l(u)\rangle\langle \phi_l(y), \phi_l(v)\rangle) = \\ =&\langle R(\phi_l(x), \phi_l(y))\phi_l(u), \phi_l(v)\rangle\end{aligned}$$

(because ϕ_l is an isometry). Thus $f : V \to f(V) =: \tilde{V}$ is a local isometry and $df_p = \iota$. Up to restricting V and $\tilde{V}$, we can assume f to be an isometry. □

As a direct consequence of this corollary, choosing $M = \tilde{M}$, we get the following.

Corollary 1.4. *Let M be an n-dimensional Riemannian manifold of constant curvature κ and $p, q \in M$ a point. Suppose we have two orthonormal bases $\{e_j\}_{j=1,\dots,n}$ of T_pM and $\{f_j\}_{j=1,\dots,n}$ of T_qM. Then there exists a neighborhood U of p, a neighborhood V of q and a map $g : U \to V$ such that $dg_p(e_j) = f_j$.*

2 Classification of constant curvature manifolds

Before proceeding, we are going to state and prove a lemma that we will need in the main theorem of today.

Lemma 2.1. *Let M, N be two Riemannian manifolds, M connected and $f_1, f_2 : M \to N$ two local isometries. If there exists a point $p \in M$ such that $f_1(p) = f_2(p)$ and $(df_1)_p = (df_2)_p$, then $f_1 \equiv f_2$.*

Proof. Let V be a neighborhood of p such that the restrictions $f_1|_V, f_2|_V$ are diffeomorphisms. For any $q \in V$, there exists a unique $v \in T_pM$ such that $\exp_p(v) = q$. Thus, since $f_i|_V$ are local isometries and diffeomorphisms, they are isometries and

$$\begin{aligned} f_2(q) =& f_2(\exp_p(v)) = \exp_{f_2(p)}((df_2)_p(v)) = \exp_{f_1(p)}((df_1)_p(v)) = f_1(\exp_p(v)) = \\ =& f_1(q) \end{aligned}$$

and since q was arbitrary, $f_1|_V \equiv f_2|_V$.

Now, since M is connected, for any $r \in M$ there exists a path $\alpha : [0,1] \to M$ such that $\alpha(0) = p, \alpha(1) = r$. Consider the set

$$A = \{t \in [0,1] |\ f_1(\alpha(t)) = f_2(\alpha(t)) \text{ and } (df_1)_{\alpha(t)} = (df_2)_{\alpha(t)}\}.$$

By what we just proved, since $V \setminus \{p\} \neq \emptyset$, there will exists a $t_0 \in (0,1]$ such that $\alpha(t_0) \in V$ and $t_0 \in A$. Thus, $0 < t_0 \leq \sup A$. Suppose by contradiction that $\sup A < 1$. If $\sup A = t_1 < 1$, we can repeat the argument about the existence of a neighborhood V of p to a neighborhood of $\alpha(t_1)$, and get a contradiction. Thus, $\sup A = 1$, so $f_1(r) = f_2(r)$, for all $r \in M$. □

Another result we will need in the proof of the next theorem is the following.

Lemma 2.2. *Let M be a complete Riemannian manifold and let $f : M \to N$ be a surjective local diffeomorphism onto a Riemannian manifold N with the following property: for all $p \in M, v \in T_pM$, $|df_p(v)| \geq |v|$. Then f is a covering map.*

Proof. Check *Lemma 3.3, Chapter 7, M. P. do Carmo, Riemannian Geometry.* □

We can now take a look at the main theorem of this lecture. Before doing that, note that by the first remark of today 0.1, we can assume that the constant curvature of the manifold will be $-1 = \kappa(\mathbb{H}^n)$,$0 = \kappa(\mathbb{R}^n)$ or $1 = \kappa(\mathbb{S}^n)$.

Theorem 2.3. *Any complete Riemannian manifold M with constant curvature κ has its universal covering space $\tilde{M}$, with the covering metric, isometric to:*

1. $\mathbb{H}^n$ *if* $\kappa = -1$;
2. $\mathbb{R}^n$ *if* $\kappa = 0$;
3. $\mathbb{S}^n$ *if* $\kappa = 1$.

Proof. Since $\tilde{M}$ is the universal cover, it is simply connected. Moreover, it is complete Riemannian and since it has the covering metric, its sectional curvature is κ.

Note that in $\mathbb{H}^n$ and $\mathbb{R}^n$, the exponential map is well defined everywhere, whereas in $\mathbb{S}^n$ is only a local diffeomorphism. Thus we are going to divide the first two cases from the third.

Denote by Δ $\mathbb{H}^n$ as well as $\mathbb{R}^n$. Pick two points $p \in \Delta$, $\tilde{p} \in \tilde{M}$ and a linear isometry $\iota : T_p\Delta \to T_{\tilde{p}}\tilde{M}$. The map $f = \exp_{\tilde{p}} \circ \iota \circ \exp_p : \Delta \to \tilde{M}$ is well defined (Δ and $\tilde{M}$ are complete with non-positive curvature). From the first corollary of Cartan's Theorem 1.3, f is a local isometry. By Lemma 2.2, since for p $df_p = \iota$ is an isometry and thus $|df_p(v)| = |v| \geq |v|$ and for $q \neq p$ $|df_p(v)| = |\tilde{J}(l)| = |J(l)| = |v| \geq |v|$, f is a covering map. Thus $\pi \circ f$ is a covering map of M (where π is the universal covering). Since $\tilde{M}$ is the universal cover, by the universal property there exists a unique homeomorphism $g : \tilde{M} \to \Delta$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\exists! g}{\dashrightarrow} & \\ \tilde{M} & \xleftarrow{\;f\;} & \Delta \\ {\scriptstyle\pi}\downarrow & \swarrow_{\pi\circ f} & \\ M & & \end{array}.$$

Thus f is a diffeomorphism and g is its inverse. Hence, f is an isometry.

Let's prove the third case. As before, take $p \in \Delta$, $\tilde{p} \in \tilde{M}$ and a linear isometry $\iota : T_p\Delta \to T_{\tilde{p}}\tilde{M}$. Let q be the antipodal point of p. Then $f = \exp_{\tilde{p}} \circ \iota \circ \exp_p : \mathbb{S}^n \backslash \{q\} \to \tilde{M}$ is well defined and by Cartan's theorem it is a local isometry. Similarly, for $p' \neq p, q$, $\tilde{p}' = f(p')$, q' its antipodal point, $f' = \exp_{\tilde{p}'} \circ \iota' \circ \exp_{p'} : \mathbb{S}^n \setminus \{q'\} \to \tilde{M}$, where $\iota' = df_{p'}$, is well defined and by Cartan's theorem it is a local isometry. Note that $\mathbb{S}^n \setminus \{q, q'\} =: W$ is connected, $f(p') = \tilde{p}' = f'(p')$ and $df_{p'} = df'_{p'}$, and hence Lemma 2.1 holds, so $f|_W \equiv f'|_W$. We can thus define the map $g : \mathbb{S}^n \to \tilde{M}$

$$g(r) = \begin{cases} f(r) & r \in \mathbb{S}^n \setminus \{q\} \\ f'(r) & r \in \mathbb{S}^n \setminus \{q'\} \end{cases}.$$

This is a gluing of local isometries, so it's a local isometry, hence a local diffeomorphism. Moreover, since $\mathbb{S}^n$ is compact, g must be a covering map, and since $\tilde{M}$ is simply connected (thus $\tilde{M}$ is its own universal cover), g must be a diffeomorphism. Thus g is an isometry. □

3 Space forms

Now that we have shown that every complete Riemannian manifold with constant curvature (that are called *space forms*) has $\mathbb{H}^n, \mathbb{R}^n$ or $\mathbb{S}^n$ as universal covering, we want to give an idea about how to get all this possible such manifolds. To do so, we first need to recall some information about group actions, in order to move our problem to a problem in group theory.

Given a group G and a space X, we say that G *acts (from the left)* on X if there exists a map

$$\begin{aligned} \cdot : G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that $1 \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for $g_1, g_2 \in G, x \in X$. We say that this action is *free* if $g \cdot x = x$ iff $g = 1$ and that this action is *transitive* if for every $x_1, x_2 \in X$, there exists a $g \in G$ such that $x_1 = g \cdot x_2$. We can restate this property as $\forall x \in X$, $Gx = X$, where we denoted by Gx the *orbit* of x, which is the set $\{g \cdot x | \ g \in G\}$. The set of all the orbits will be denoted by X/G, and if X is a topological space, it can be endowed with the quotient topology given by the natural projection

$$\begin{aligned} \pi : X &\to X/G \\ x &\mapsto Gx \end{aligned}.$$

It can be useful, in the case of X a topological space, to take G to be the group of homeomorphisms. Similarly, if X is differentiable, we can consider G the group of diffeomorphisms.

Now, consider M a topological space and G a group (in this case we can take it to be the group of homeomorphisms). We say that G *acts in a totally discontinuous manner* on M if for every point $x \in M$, there exists a neighborhood U of x such that $g(U) \cap U = \emptyset$ for every $g \in G \setminus \{1\}$. It can be proven that π is a regular covering map (i.e. $\pi_*(\pi_1(\tilde{M}, \tilde{p}))$ is a normal subgroup of $\pi_1(M, \pi(\tilde{p}))$) and G is the group of covering transformations (given two coverings $p_i : Y_i \to X$ a covering transformation is a continuous map $F : Y_1 \to Y_2$ such that $p_1 = p_2 \circ F$).

If M is a Riemannian manifold and Γ a subgroup of the group of isometries of M that acts in a totally discontinuous manner, then M/Γ has a differentiable structure such that $\pi : M \to M/\Gamma$ is a local diffeomorphism. In addition, we can define a Riemannian metric on M/Γ in the following way: for $p \in M/\Gamma$, $\tilde{p} \in \pi^{-1}(p)$ and for any $u, v \in T_p(M/\Gamma)$,

$$\langle u, v \rangle_p := \langle d\pi^{-1}(u), d\pi^{-1}(v) \rangle_{\tilde{p}}.$$

With this metric, π is obviously a local isometry.

It is well defined. Indeed, by regularity of the covering map π, we get that the action of Γ on $\pi^{-1}(p)$ is transitive, and thus for every $\tilde{q} \in \pi^{-1}(p)$ there exists a $\gamma \in \Gamma$ such that $\gamma \cdot \tilde{p} = \tilde{q}$. This implies that the definition above does not depend on the choice of $\tilde{p}$. This metric is called the *metric on M/Γ induced by the covering π.*

We have that M/Γ is complete if and only if M is complete, and it has constant sectional curvature if and only if M has constant sectional curvature.

With the following theorem we want to show that taking M to be $\mathbb{H}^n, \mathbb{R}^n$ or $\mathbb{S}^n$ (depending on the sign of the constant curvature), we get all the possible space forms.

Proposition 3.1. *Let M be a complete Riemannian manifold of constant sectional curvature $\kappa \in \{-1, 0, 1\}$. Let $\tilde{M}$ be the universal covering of M. Then M is isometric to $\tilde{M}/\Gamma$, where $\tilde{M}$ is $\mathbb{H}^n$ if $\kappa = -1$, $\mathbb{R}^n$ if $\kappa = 0$ and $\mathbb{S}^n$ if $\kappa = 1$, and Γ is a subgroup of the group of isometries of $\tilde{M}$ which acts in a totally discontinuous manner on $\tilde{M}$, and the metric on $\tilde{M}/\Gamma$ is induced by the covering $\pi : \tilde{M} \to \tilde{M}/\Gamma$.*

Proof. Let $p : \tilde{M} \to M$ be the universal covering of M, and provide $\tilde{M}$ with the covering metric. With the covering metric, p is a local isometry. Let Γ be the group of covering transformations of p. Then Γ is a subgroup of the group of isometries of $\tilde{M}$ which acts in a totally discontinuous manner on $\tilde{M}$. Thus, we can induce on $\tilde{M}/\Gamma$ the Riemannian metric on $\tilde{M}/\Gamma$ induced by the covering $\pi : \tilde{M} \to \tilde{M}/\Gamma$.

Since p is regular, we have that $p(\tilde{x}) = p(\tilde{y})$ for $\tilde{x}, \tilde{y} \in \tilde{M}$ if and only if $\pi(\tilde{x}) = \Gamma\tilde{x} = \Gamma\tilde{y} = \pi(\tilde{y})$. Therefore the equivalence classes given by p and π on $\tilde{M}$ are the same, and this implies that we have a bijection $\xi : M \to \tilde{M}/\Gamma$, such that $\pi = \xi \circ p$. Since π and p are local isometries, ξ must be a local isometry, and since it is a bijection, it must be an isometry. □

Therefore, we can see how the problem of finding all the space forms is equivalent to the group theory problem of determining all the possible subgroups Γ of the group of isometries.

Two nice results coming from this classification are the following.

Proposition 3.2. *Let M^{2n} be a space form of sectional curvature $\kappa = 1$. Then $M \cong \mathbb{S}^{2n}$ or $M \cong \mathbb{RP}^{2n}$.*

Proof. See Exercise 3. □

Proposition 3.3. *Let M^2 be compact orientable of genus $g > 1$. Then this can be provided with a metric of constant negative curvature $\kappa = -1$.*

Proof. See *Proposition 4.5, Chapter 8, M. P. do Carmo, Riemannian Geometry.* □

4 Exercises

1. Show that $\mathbb{H}^n$, with metric $g_{i,j}(x_1 \ldots, x_n) = \frac{\delta_{i,j}}{x_n^2}$, has constant sectional curvature equal to -1.

2. Show that if G acts in a totally discontinuous manner, then the projection $\pi : M \to M/G$ is a regular covering map, and G is the group of covering transformations.

3. Prove Proposition 3.2.