# Rauch comparison theorem

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*Notation:* We will use the notation of the reference M. P. do Carmo, *Riemannian Geometry*. In particular the curvature tensor has opposite sign and the covariant derivative of a vector field $V$ defined along a curve $c$ will simply be denoted $V'$ instead of $\frac{\nabla}{dt}V$. So, for instance, with our notation the Jacobi equation is written as

$$J'' + R(\gamma', J)\gamma' = 0.$$

## Motivation behind the theorem

Throughout we use $M$ to denote a $n$-dimensional Riemannian manifold. Recall the following result:

**Proposition.** *If $\gamma : [0, l] \to M$ is a normalized geodesic, i.e., parametrized by arc-length, $J$ is a Jacobi field along $\gamma$ with $J(0) = 0$ and $\langle \gamma'(0), J'(0)\rangle = 0$, then as $t \to 0$*

$$|J(t)| = |J'(0)| \left( t - \frac{1}{6}Kt^3 + o(t^3) \right),$$

*where $K$ is the sectional curvature of the plane generated by $\gamma'(0)$ and $J'(0)$.*

Hence, for small times, the smaller $K$, the large $|J(t)|$ will be. Consider now another triple $\tilde{M}, \tilde{\gamma}, \tilde{J}$ with the same conditions as above. Then,

$$\tilde{K} \geq K \text{ implies } |\tilde{J}(t)| \leq |J(t)|, \text{ for small } t.$$

The Rauch theorem provides the conditions when this inequality holds <u>for all times</u>. It allows to compare the Jacobi fields on different manifolds if one sectional curvature dominates the other.

*Proof of Proposition.* Expand $|J(t)|^2$ into Taylor series around the origin.

$$\langle J(t), J(t)\rangle = \sum_{n=0}^{4} \langle J, J\rangle^{(n)}(0)\frac{t^n}{n!} + o(t^4).$$

Computing the derivatives we take into account the initial condition $J(0) = 0$ and the Jacobi equation $J''(t) + R(\gamma'(t), J(t))\gamma'(t) = 0$ which implies $J''(0) = 0$.

$$\begin{aligned}
&\langle J, J\rangle^{(1)}(0) = 2\langle J, J'\rangle|_{t=0} = 0 \\
&\langle J, J\rangle^{(2)}(0) = 2\left(\langle J, J''\rangle + \langle J', J'\rangle\right)|_{t=0} = 2|J'(0)|^2 \\
&\langle J, J\rangle^{(3)}(0) = 2\left(\langle J, J'''\rangle + 3\langle J', J''\rangle\right)|_{t=0} = 0 \\
&\langle J, J\rangle^{(4)}(0) = 2\left(\langle J, J^{(4)}\rangle + 4\langle J', J'''\rangle + 3|J''|^2|\right)|_{t=0} = 8\langle J', J'''\rangle(0)
\end{aligned}$$

Now it is left to compute $J'''$. For any field $W$ we have

$$\begin{aligned}
&\frac{d}{dt}\langle R(\gamma', W)\gamma', J\rangle = \langle (R(\gamma', W)\gamma')', J\rangle + \langle R(\gamma', W)\gamma', J'\rangle \\
&\qquad\| \\
&\frac{d}{dt}\langle R(\gamma', J)\gamma', W\rangle = \langle \underbrace{(R(\gamma', J)\gamma')'}_{J'''}, W\rangle + \langle R(\gamma', J)\gamma', W'\rangle
\end{aligned}$$

Subtracting one from the other and evaluating at $t = 0$ we obtain

$$\langle J''', W\rangle(0) = -\langle R(\gamma', W)\gamma', J'\rangle(0) = -\langle R(\gamma', J')\gamma', W\rangle(0),$$

so that

$$J'''(0) = R(\gamma', J')\gamma'|_{t=0}.$$

The $4^{\text{th}}$ derivative becomes

$$\langle J, J\rangle^{(4)}(0) = -8\langle J', R(\gamma', J')\gamma'\rangle(0) = -8K|J'(0)|^2,$$

where $K$ is the sectional curvature build on vectors $\gamma'(0)$ and $J'(0)$. Therefore,

$$|J(t)|^2 = |J'(0)|^2\left(t^2 - \frac{K}{3}t^4\right) + o(t^4)$$

or

$$|J(t)| = t - \frac{K}{6}t^3 + o(t^3).$$

□

# The index lemma

Let $V$ be a piecewise differentiable vector field along a geodesic $\gamma : [0, a] \to M$. For all $t_0 \in [0, a]$, define

$$I_{t_0}(V, V) = \int_0^{t_0} \left(\langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle\right) dt.$$

Let $V_0 \in T_{\gamma_{t_0}}M$ and consider all piecewise differentiable fields $V$ along $\gamma$ with prescribed values $V(0) = 0$ and $V(t_0) = V_0$. Provided that $\gamma(t_0)$ is not conjugate to $\gamma(0)$, there exists a unique Jacobi field $J$ along $\gamma$ with the same prescribed values $J(0) = 0$ and $J(t_0) = V_{t_0}$. With additional assumption that there are no conjugate points in the whole interval $(0, t_0)$, the index lemma asserts that this Jacobi field $J$ minimises the expression above.

**Lemma.** *Let*
- $\gamma : [0, a] \to M$ *be a geodesic without conjugate points to* $\gamma(0)$ *on* $(0, a]$*;*
- $J$ *be a Jacobi field along* $\gamma$*, with* $\langle J, \gamma' \rangle = 0$*;*
- $V$ *be a piecewise differentiable vector field along* $\gamma$*, with* $\langle V, \gamma' \rangle = 0$*;*
- $J(0) = V(0) = 0$ *and* $J(t_0) = V(t_0)$ *for some* $t_0 \in (0, a]$.

*Then*

$$I_{t_0}(J, J) \leq I_{t_0}(V, V)$$

*and equality occurs if and only if* $V = J$ *on* $[0, t_0]$.

*Proof.* The vector space of all Jacobi fields $J$ along $\gamma$ with $J(0) = 0$ and $\langle J, \gamma' \rangle = 0$ has dimension $n-1$, where $n = \dim M$. Let $\{J_1, ..., J_{n-1}\}$ be any collection of Jacobi fields that form a basis for that space, so that

$$J = \sum_{i=1}^{n-1} \alpha_i J_i,$$

where $\{\alpha_i\}_{i=1}^{n-1}$ are constants.

Since there are no conjugate points on $(0, a]$, the collection $\{J_1(t), ..., J_{n-1}(t)\}$ forms a basis in $\gamma'(t)^{\perp} \subset T_{\gamma(t)}M$, i.e., the subspace orthogonal to $\gamma'(t)$. Therefore, for $t \neq 0$, we can decompose our vector field $V$ as

$$V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t),$$

where $f_i$ are piecewise differentiable functions on $(0, a]$.

Now we are going to rewrite the expression for the integrand in the definition of $I_{t_0}$ via $\{f_i\}$. Working on the interior of each subinterval where $V$ is differentiable we will obtain the following identity:

$$\langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle = \left| \sum_i f_i' J_i \right|^2 + \frac{d}{dt} \langle \sum_i f_i J_i, \sum_j f_j J_j' \rangle. \tag{1}$$

The first term on the left-hand side of (1) becomes

$$\begin{aligned} \langle V', V' \rangle = &\langle \sum_i f_i' J_i, \sum_j f_j' J_j \rangle + \langle \sum_i f_i' J_i, \sum_j f_j J_j' \rangle \\ &+ \langle \sum_i f_i J_i', \sum_j f_j' J_j \rangle + \langle \sum_i f_i J_i', \sum_j f_j J_j' \rangle \end{aligned}$$

For the second term the Jacobi equation yields

$$R(\gamma', V)\gamma' = R(\gamma', \sum_i f_i J_i)\gamma' = \sum_i f_i R(\gamma', J_i)\gamma' = -\sum_i f_i J_i'',$$

so that

$$\langle R(\gamma', V)\gamma', V\rangle = -\langle \sum_i f_i J_i'', \sum_j f_j J_j\rangle,$$

and together the left-hand side reads

$$\begin{aligned}\langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle = \left|\sum_i f_i' J_i,\right|^2 &+ \langle \sum_i f_i' J_i, \sum_j f_j J_j'\rangle + \langle \sum_i f_i J_i', \sum_j f_j' J_j\rangle \\ &+ \langle \sum_i f_i J_i', \sum_j f_j J_j'\rangle + \langle \sum_i f_i J_i'', \sum_j f_j J_j\rangle.\end{aligned}$$

The derivative of the right-hand side of (1) is

$$\begin{aligned}\frac{d}{dt}\langle \sum_i f_i J_i, \sum_j f_j J_j'\rangle = \langle \sum_i f_i' J_i, \sum_j f_j J_j'\rangle &+ \langle \sum_i f_i J_i, \sum_j f_j' J_j'\rangle \\ &+ \langle \sum_i f_i J_i', \sum_j f_j J_j'\rangle + \langle \sum_i f_i J_i, \sum_j f_j J_j''\rangle.\end{aligned}$$

It is left to show that the blue terms are the same. Define

$$h(t) = \langle J_i', J_j\rangle - \langle J_i, J_j'\rangle.$$

Note that $h(0) = 0$ and, using Jacobi equation,

$$\begin{aligned}h'(t) &= \langle J_i'', J_j\rangle + \langle J_i', J_j'\rangle - \langle J_i', J_j'\rangle - \langle J_i, J_j''\rangle \\ &= -\langle R(\gamma', J_i)\gamma', J_j\rangle + \langle J_i, R(\gamma', J_j)\gamma'\rangle.\end{aligned}$$

It follows that $h'(t) = 0$ from the permutation-of-arguments property of the curvature. Hence, $h(t) \equiv 0$ and the claim follows by the distributivity of the curvature, which concludes the proof of (1).

Therefore, integrating over $t$ we obtain

$$I_{t_0}(V, V) = \langle \sum_i f_i J_i, \sum_j f_j J_j'\rangle(t_0) + \int_0^{t_0} \left|\sum_i f_i' J_i\right|^2 dt,$$

and for the Jacobi field

$$I_{t_0}(J, J) = \langle \sum_i \alpha_i J_i, \sum_j \alpha_j J_j'\rangle(t_0).$$

The condition $J(t_0) = V(t_0)$ forces $\alpha_i = f_i(t_0)$ which yields

$$I_{t_0}(V, V) - I_{t_0}(J, J) = \int_0^{t_0} \left|\sum_i f_i' J_i\right|^2 dt \geq 0.$$

This proves the first part of the lemma.

If $I_{t_0}(V,V) = I_{t_0}(J,J)$, then $\sum_i f_i' J_i = 0$, which implies $f_i' = 0$, for all $t \in (0, t_0]$, by the linear independence of $\{J_i\}$. Hence, $f_i(t) = f_i(t_0) = \alpha_i$, i.e., $V = J$.

$\square$

## Rauch comparison theorem

**Theorem.** *(Rauch) Let $(M, g)$ and $(\tilde{M}, \tilde{g})$ be $n$- respectively $(n+k)$-dimensional Riemannian manifolds ($k \geq 0$). Let $\gamma : [0, a] \to M$ and $\tilde{\gamma} : [0, a] \to \tilde{M}$, $k \geq 0$, be geodesics such that $|\gamma'(t)| = |\tilde{\gamma}'(t)|$. Let $J$ and $\tilde{J}$ be Jacobi fields along $\gamma$ and $\tilde{\gamma}$, respectively, such that*

$$J(0) = 0, \;\; \tilde{J}(0) = 0,$$
$$\langle J'(0), \gamma'(0)\rangle = \langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle,$$
$$|J'(0)| = |\tilde{J}'(0)|.$$

*Assume that $\tilde{\gamma}$ does not have conjugate points on $(0, a]$ and that for all $t \in [0, a]$, $x \in T_{\gamma(t)}M$, and $\tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$ (not parallel to $\gamma'$ resp. $\tilde{\gamma}'$), we have*

$$K(\gamma'(t), x) \leq \tilde{K}(\tilde{\gamma}'(t), \tilde{x}).$$

*Then,*

$$|\tilde{J}(t)| \leq |J(t)| \text{ for all } t \in [0, a].$$

*Moreover, if for some $t_0 \in (0, a]$, $|\tilde{J}(t_0)| = |J(t_0)|$, then $K(\gamma'(t), J(t)) = \tilde{K}(\tilde{\gamma}'(t), \tilde{J}(t))$ for all $t \in (0, t_0]$.*

*Proof.* From Lemma 1.4.7 we have that

$$J^{\|}(t) = \frac{t}{|\gamma'|^2}\langle J'(0), \gamma'(0)\rangle\gamma'(t) + \frac{1}{|\gamma'|^2}\langle J(0), \gamma'(0)\rangle\gamma'(t),$$

$$\tilde{J}^{\|}(t) = \frac{t}{|\tilde{\gamma}'|^2}\langle \tilde{J}'(0), \tilde{\gamma}'(0)\rangle\tilde{\gamma}'(t) + \frac{1}{|\tilde{\gamma}'|^2}\langle \tilde{J}(0), \tilde{\gamma}'(0)\rangle\tilde{\gamma}'(t).$$

Therefore, $|J^{\|}(t)|^2 = |\tilde{J}^{\|}(t)|^2$, so we may assume that $\langle J(t), \gamma'(t)\rangle = \langle \tilde{J}(t), \tilde{\gamma}'(t)\rangle = 0$. We may also assume that $|J'(0)| = |\tilde{J}'(0)| \neq 0$, for otherwise $J \equiv 0 \equiv \tilde{J}$.

Define $v(t) := |J(t)|^2$, and $\tilde{v}(t) := |\tilde{J}(t)|^2$. Since $\tilde{\gamma}$ has no conjugate point on $(0, a]$ and $\tilde{J} \not\equiv 0$, we have $\tilde{v}(t) \neq 0$ on $(0, a]$. Thus, $\frac{v(t)}{\tilde{v}(t)}$ is well-defined for all $t \in (0, a]$ and using L'Hospital's rule we see that

$$\lim_{t\to 0+} \frac{v(t)}{\tilde{v}(t)} = \lim_{t\to 0+} \frac{\langle J(t), J(t)\rangle}{\langle \tilde{J}(t), \tilde{J}(t)\rangle} = \lim_{t\to 0+} \frac{\langle J(t), J(t)\rangle'}{\langle \tilde{J}(t), \tilde{J}(t)\rangle'} = \lim_{t\to 0+} \frac{2\langle J'(t), J(t)\rangle}{2\langle \tilde{J}'(t), \tilde{J}(t)\rangle}$$

$$= \lim_{t\to 0+} \frac{\langle J''(t), J(t)\rangle + \langle J'(t), J'(t)\rangle}{\langle \tilde{J}''(t), \tilde{J}(t)\rangle + \langle \tilde{J}'(t), \tilde{J}'(t)\rangle} = \frac{|J'(0)|^2}{|\tilde{J}'(0)|^2} = 1.$$

So, if we show that $\frac{d}{dt}(\frac{v(t)}{\tilde{v}(t)}) \geq 0$ for all $t \in (0, a]$, or equivalently

$$v'(t)\tilde{v}(t) \geq v(t)\tilde{v}'(t) \tag{2}$$

for all $t \in (0, a]$, it will follow that $|\tilde{J}(t)| \leq |J(t)|$ for all $t \in [0, a]$.

Fix $t_0 \in (0, a]$. If $v(t_0) = 0$ then $v'(t_0) = 2\langle J'(t_0), J(t_0)\rangle = 0$ so (2) is satisfied. If not, then define

$$U(t) := \frac{1}{\sqrt{v(t_0)}} J(t), \ \tilde{U}(t) := \frac{1}{\sqrt{\tilde{v}(t_0)}} \tilde{J}(t).$$

Then,

$$\begin{aligned}
\frac{v'(t_0)}{v(t_0)} &= \frac{2\langle J'(t_0), J(t_0)\rangle}{\langle J(t_0), J(t_0)\rangle} = 2\langle U'(t_0), U(t_0)\rangle = 2\int_0^{t_0} \langle U', U\rangle' dt \\
&= 2\int_0^{t_0} \big(\langle U', U'\rangle + \langle U'', U\rangle\big) dt = 2\int_0^{t_0} \big(\langle U', U'\rangle - \langle R(\gamma', U)\gamma', U\rangle\big) dt \\
&= 2I_{t_0}(U, U).
\end{aligned}$$

Similarly, $\frac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = 2I_{t_0}(\tilde{U}, \tilde{U})$. Therefore, if we show $I_{t_0}(\tilde{U}, \tilde{U}) \leq I_{t_0}(U, U)$, (2) follows.

Let $\{e_i\}_{i=1}^n$ and $\{\tilde{e}_i\}_{i=1}^{n+k}$ be parallel orthonormal bases along $\gamma$ and $\tilde{\gamma}$, respectively, such that

$$\begin{aligned}
&e_1(t) = \gamma'(t)/|\gamma'(t)|, \ e_2(t_0) = U(t_0), \\
&\tilde{e}_1(t) = \tilde{\gamma}'(t)/|\tilde{\gamma}'(t)|, \ \tilde{e}_2(t_0) = \tilde{U}(t_0).
\end{aligned}$$

For each vector field $V(t) = \sum_{i=1}^n g_i(t)e_i(t)$ along $\gamma$ we define a vector field $\phi V$ along $\tilde{\gamma}$ by $(\phi V)(t) = \sum_{i=1}^n g_i(t)\tilde{e}_i(t)$. For any $V_1, V_2$ we have

$$\langle \phi V_1, \phi V_2\rangle(t) = \sum_{i=1}^n g_{1,i}(t)g_{2,i}(t)(t) = \langle V_1, V_2\rangle,$$

since the bases are orthonormal. Moreover, since the bases are parallel

$$(\phi V)' = \sum_{i=1}^n g_i'(t)\tilde{e}_i(t) = \phi(V').$$

By the assumption on the sectional curvature we therefore have

$$\langle U, R(\gamma', U)\gamma'\rangle = \frac{1}{|\gamma'|^2|U|^2} K(U, \gamma') \leq \frac{1}{|\tilde{\gamma}'|^2|\phi U|^2} \tilde{K}(\phi U, \tilde{\gamma}') = \langle \phi U, \tilde{R}(\tilde{\gamma}', \phi U)\tilde{\gamma}'\rangle.$$

As a consequence,

$$\begin{aligned}
I_{t_0}(\phi U, \phi U) &= \int_0^{t_0} \big(\langle (\phi U)', (\phi U)'\rangle - \langle \phi U, \tilde{R}(\tilde{\gamma}', \phi U)\tilde{\gamma}'\rangle\big) dt \\
\leq \int_0^{t_0} &\big(\langle U', U'\rangle - \langle U, R(\gamma', U)\gamma'\rangle\big) dt = I_{t_0}(U, U).
\end{aligned}$$

Furthermore, $\tilde{U}$ and $\phi U$ are both vector fields along $\tilde{\gamma}$ (which by assumption does not have conjugate points on $(0, a]$), and $\tilde{U}$ is a Jacobi field and $\phi U$ is differentiable, both are orthogonal to $\gamma'$ and satisfy $\tilde{U}(0) = \phi U(0) = 0$, $\tilde{U}(t_0) = \phi U(t_0) = \tilde{e}_2(t_0)$. Therefore, by the index lemma,

$$I_{t_0}(\tilde{U}, \tilde{U}) \leq I_{t_0}(\phi U, \phi U).$$

Since, $t_0 \in (0, a]$ was arbitrary this shows that (2) holds for all $t \in (0, a]$, as desired.

It only remains to show the last part of the theorem. Suppose $t_0 \in (0, a]$ is such that $|\tilde{J}(t_0)| = |J(t_0)|$. Then, based on what we have already shown, $\frac{d}{dt}(\frac{v(t)}{\tilde{v}(t)}) = 0$ for all $t \in (0, t_0]$, or equivalently $v'(t)\tilde{v}(t) = v(t)\tilde{v}'(t)$ for all $t \in (0, t_0]$. But this shows that

$$I_t(\tilde{U}, \tilde{U}) = I_t(\phi U, \phi U) = I_t(U, U)$$

for all $t \in (0, t_0]$. From the first equality it follows that $\tilde{U} = \phi U$ for all $t \in [0, t_0]$ (using the index lemma). And from the second it follows that

$$\langle U, R(U, \gamma')\gamma' \rangle(t) = \langle \phi U, \tilde{R}(\phi U, \tilde{\gamma}')\tilde{\gamma}' \rangle(t)$$

for all $t \in (0, t_0]$. But then

$$K(J(t), \gamma'(t)) = K(U(t), \gamma'(t)) = \tilde{K}(\tilde{U}(t), \tilde{\gamma}'(t)) = \tilde{K}(\tilde{J}(t), \tilde{\gamma}'(t)),$$

for all $t \in (0, t_0]$. □

## Corollary

**Corollary.** *Suppose that the sectional curvature $K$ of a manifold $M$ satisfies*

$$L \leq K \leq H$$

*for some positive constants $L, H$. Then the distance $d$ between two consecutive conjugate points along a geodesic on $M$ satisfies*

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

*Proof.* Let $p$ be any point on $M$ and $\gamma$ be a unit-speed geodesic with $\gamma(0) = p$. Let $J$ be a Jacobi field along $\gamma$ with $J(0) = 0$, $\langle J, \gamma' \rangle$ and $|J'(0)| = 1$

For the lower bound on $d$ we compare $M$ with the $n$-dimensional sphere of curvature $H$, $S^n(H)$. Note that $S^n(H)$ has distance $\pi/\sqrt{H}$ between its conjugate points. Let $\tilde{\gamma} : [0, \infty) \to S^n(H)$ be a unit-speed geodesic on $S^2(H)$ and let $\tilde{J}$ be a Jacobi field along $\tilde{\gamma}$ with $\tilde{J}(0) = 0$, $\langle \tilde{J}, \tilde{\gamma} \rangle = 0$ and $|\tilde{J}'(0)| = 1$. Then the Rauch comparison theorem applies on the interval $t \in [0, \pi/\sqrt{H})$, implying that $0 < |\tilde{J}(t)| \leq |J(t)|$ for all $t \in (0, \pi/\sqrt{H})$. Therefore, since $\gamma$ and $J$ were chosen arbitrarily (up to scale), it follows that the distance between conjugate points $d \geq \pi/\sqrt{H}$.

For the upper bound on $d$ we compare $M$ with the $n$-dimensional sphere of curvature $L$, $S^n(L)$. Let $\tilde{\gamma}$ and $\tilde{J}$ be analogous to $\tilde{\gamma}$ and $\tilde{J}$ above. Suppose that $d > \pi/\sqrt{L}$, so that $\gamma$ has no conjugate points on $[0, \pi/\sqrt{L}]$. Then the Rauch comparison theorem applies (with the roles of $M$ and $\tilde{M}$ swapped). This shows that

$$0 < |J(t)| \leq |\tilde{J}(t)|$$

for all $t \in [0, \pi/\sqrt{L}]$. But $\tilde{\gamma}$ has a conjugate point of dimension $n-1$ at $t = \pi/\sqrt{L}$, so $\tilde{J}(\pi/\sqrt{L}) = 0$, a contradiction. We deduce that $d \leq \pi/\sqrt{L}$. □

# Exercises

**Problem 1.** Let $(M, g)$ be a complete Riemannian manifold with non-positive sectional curvature. Use Rauch comparison theorem to show that for any $p \in M$, $X_p \in T_pM$ and $Y_p \in T_pM = T_{X_p}(T_pM)$ we have

$$|(d\exp_p)_{X_p}(Y_p)| \geq |Y_p|.$$

Conclude that for any (differentiable) curve $c : [0, a] \to T_pM$ it follows that

$$L(c) \leq L(\exp_p \circ c).$$

**Problem 2.** Let $(M, g)$ be a complete Riemannian manifold with non-positive sectional curvature and consider a geodesic triangle in $M$ with side lengths $a$, $b$ and $c$ with opposite angles $A$, $B$ and $C$ respectively.

1. Show that $a^2 + b^2 - 2ab\cos C \leq c^2$.
2. Show that $A + B + C \leq \pi$.

Hint: Use Problem 1.

**Problem 3.** Fix points $p \in M$, $\tilde{p} \in \tilde{M}$ and isometry $I : T_pM \to T_{\tilde{p}}\tilde{M}$. Consider in $T_pM$ a piece-wise smooth path $v[0, a] \to T_pM$ and let $\gamma = \exp_p(v)$, $\tilde{\gamma} = \exp_{\tilde{p}}(I \circ v)$. Suppose that for every $s \in [0, a]$ the geodesic $t \to \exp_p(tv(s))$, $0 \leq t \leq 1$, does not have conjugate points with respect to $p$. Then, if for every sectional curvatures we have $K(p, \sigma) \geq K(\tilde{p}, \tilde{\sigma})$, show the following relation between the lengths of curves holds:

$$L(\gamma) \leq L(\tilde{\gamma}).$$

Remark: For simplicity assume that $v$ is always non-zero. Note also that $M$ here plays the role of $\tilde{M}$ in the formulation of Rauch theorem above.

Hint: Consider the geodesic variation $\sigma(t, s) = \exp_p(tv(s))$ and the Jacobi field $t \to J_s(t) = \frac{\partial}{\partial s}\sigma(t, s)$. Apply the Rauch theorem to $J_s^\perp$ and the corresponding field $\tilde{J}_s^\perp$ on $\tilde{M}$.