VOLUME COMPARISON

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There is in general no natural notion of volume in smooth manifolds, but with the addition of a Riemannian metric this is no longer true—the key idea being that the tangent vectors at any point of the manifold generate a parallelepiped of a certain volume determined by the inner product. This observation leads to the definition of the canonical measure on Riemannian manifolds, allowing us to speak of volume in that setting.

It is natural that the volume of a ball centered around a given point of the manifold depends on the curvature of the manifold in some way, raising the question of whether suitable assumtions on the curvature can lead to upper or lower bounds for the volume. This lecture culminates in a proof of the Bishop–Günther volume comparison theorem, which answers the question in the affirmative.

This introduction to volume comparison in Riemannian geometry is based on Section 3H in the book of Gallot, Hulin and Lafontaine [1].

1 Preliminaries

The following is a consequence of Hopf–Rinow's theorem.

Proposition 1.1. *Let (M, g) be a complete Riemannian manifold, and c a geodesic with points $a < b$ in its domain.*

(i) if there exists no geodesic shorter than c from $c(a)$ to $c(b)$, then c is minimal on $[a, b]$;

(ii) if c is minimal on $[a, b]$ and there exists another geodesic of the same length as c from $c(a)$ to $c(b)$, then c is no longer minimal on any $[a, b + \varepsilon]$, $\varepsilon > 0$;

(iii) if c is minimal on an interval I, then it is also minimal on any subinterval $J \subseteq I$.

Let (M, g) be a complete Riemannian manifold and fix $p \in M$. For $v \in T_pM$, denote $c_v(t) = \exp_p tv$ and let

$$I_v = \{t \in \mathbb{R} : c_v \text{ is minimal on } [0, t]\}.$$

From Proposition 1.1 it follows that I_v is closed and of the form $I_v = [0, \rho(v)]$, where $\rho(v)$ is possibly infinite. Furthermore, the map $v \mapsto \rho(v)$ is continuous, implying that the set

$$U_p = \left\{v \in T_pM : \|v\| < \rho\left(\frac{v}{\|v\|}\right)\right\} = \left\{v \in T_pM : \rho(v) > 1\right\}$$

is an open neighborhood of 0 in T_pM, with boundary $\partial U_p = \{v \in T_pM : \rho(v) = 1\}$.

Definition 1.2. Let (M, g) be a complete Riemannian manifold and $p \in M$. The *cut-locus* of p is defined as

$$\mathrm{Cut}_p = \exp_p(\partial U_p),$$

with U_p as above.

Proposition 1.3. *For any $p \in M$, $\exp_p(U_p)$ and Cut_p are disjoint, and*

$$M = \exp_p(U_p) \cup \mathrm{Cut}_p.$$

Proof. Let $q \in M$. We know from Hopf–Rinow's theorem that there exists a minimal geodesic $c_v : [0, 1] \longrightarrow M$ from p to q. Then $\rho(v) \geqslant 1$, so v is in the closure of U_p. This proves the equality.

For the second part, suppose $q \in \exp_p(U_p) \cap \mathrm{Cut}_p$. Since $q \in \exp_p(U_p)$, there exists a geodesic c with $c(0) = p$ and $c(a) = q$ that is minimal on $[0, a + \varepsilon]$ for some $\varepsilon > 0$. But since $q \in \mathrm{Cut}_p$, there also exists a geodesic γ with $\gamma(0) = p$ and $\gamma(b) = q$ that is minimal, but no longer minimal after b. This contradicts Proposition 1.1 (ii). □

Example 1.4. Let $p \in S^n$. All geodesics in S^n are minimizing before distance π but not after it, so $\rho(v) = \pi$ for all $v \in T_pS^n$ with $\|v\| = 1$. We get

$$U_p = \{v \in T_pS^n : \|v\| < \pi\} = B(0, \pi),$$

so $\exp_p(U_p) = S^n \setminus \{-p\}$ and $\mathrm{Cut}_p = \{-p\}$.

2 Densities and the canonical measure

Definition 2.1. Let M^n be a smooth manifold with an atlas (U_i, φ_i). A *density* on M associates to each chart a measure μ_i on $\varphi_i(U_i)$ with the following properties:

(i) μ_i is absolutely continuous and has strictly positive density with respect to the Lebesgue measure;

(ii) if $U_i \cap U_j \neq \varnothing$ and f is continuous with compact support in $\varphi_i(U_i \cap U_j)$, then

$$\int_{\varphi_i(U_i \cap U_j)} f \, d\mu_i = \int_{\varphi_j(U_i \cap U_j)} \left(f \circ \varphi_i \circ \varphi_j^{-1}\right) \left| J\left(\varphi_i \circ \varphi_j^{-1}\right)\right| d\mu_j. \tag{1}$$

A density (μ_i) can be used to define a positive measure δ on the manifold M by setting

$$\delta(f) = \int_{\varphi_i(U_i)} f \circ \varphi^{-1} \, d\mu_i$$

for continuous functions f with support contained in U_i, and extending to arbitrary continuous functions using partitions of unity. The compatability condition (1) ensures that δ is well-defined. Furthermore, it is easily verified that if δ and δ' are the measures associated with two densities, then there exists a strictly positive continuous function f such that $\delta = f\delta'$.

For a Riemannian manifold (M, g), the metric suggests a natural definition for the densities of each chart (U, φ). Recall that the parallelepiped generated by the tangent vectors $\partial_1, \dots, \partial_n$ at a point $p \in U$ has volume $\sqrt{\det(g_{ij})}$, where

$$g|_U = g_{ij} dx^i \otimes dx^j$$

is the local expression of g in U.

Definition 2.2. Let (M^n, g) be a Riemannian manifold with an atlas (U_k, φ_k). The *canonical measure* v_g on M is given by the densities

$$\mu_k(A) = \int_A \sqrt{\det(g_{ij})} \circ \varphi^{-1} \, d\lambda,$$

where $A \subseteq \varphi_k(U_k)$ is any Borel set, g_{ij} are the coefficient functions of g in U_k, and λ is the Lebesgue measure on $\mathbb{R}^n$.

One can show that v_g is independent of both the atlas and the partition of unity chosen in its construction. In particular, if M is orientable, then v_g is given by a volume form.

Definition 2.3. The *volume* of a Riemannian manifold M is given by the (possibly infinite) integral

$$\mathrm{vol}(M, g) = \int_M v_g.$$

The following result makes it easier to compute volumes in practice.

Lemma 2.4. *Let M be a complete Riemannian manifold. For any $p \in M$, the cut-locus* Cut_p *has measure zero.*

Proof. Exercise. ◻

Using the exponential chart, it follows that

$$\begin{aligned}\mathrm{vol}(M, g) = \mathrm{vol}(\exp_p(U_p), g) &= \int_{U_p} \sqrt{\det(g_{ij})} \circ \exp_p \, d\lambda \\ &= \int_{S^{n-1}} \int_0^{\rho(u)} \left(\sqrt{\det(g_{ij})} \circ \exp_p \right) (tu) t^{n-1} \, dt du, \qquad (2)\end{aligned}$$

where du is the canonical measure on the unit sphere in (T_pM, g_p).

Let $c(t) = \exp_p tu$ with $u \in S^{n-1}$ and take an orthonormal basis $\{u, e_2, \dots, e_n\}$ of T_pM. We denote by Y_i the Jacobi fields satisfying

$$Y_i(0) = 0 \quad \text{and} \quad Y_i'(0) = e_i$$

(notation: $Y' = \frac{\nabla}{dt}Y$). Recall that $T_{tu}\exp_p(u) = c'(t)$ and $T_{tu}\exp_p(e_i) = \frac{1}{t}Y_i(t)$, so we have

$$(g_{ij} \circ \exp_p)(tu) = g\big(T_{tu}\exp_p(e_i), T_{tu}\exp_p(e_j)\big) = \frac{1}{t^2} g(Y_i(t), Y_j(t))$$

when $i, j > 1$, and $(g_{1i} \circ \exp_p)(tu) = \delta_{1i}$ since both $Y_i(0)$ and $Y_i'(0)$ are orthogonal to c'. Hence

$$\left(\sqrt{\det(g_{ij})} \circ \exp_p\right)(tu) = \sqrt{t^{-2(n-1)} \det(g(Y_i(t), Y_j(t)))} = \underbrace{t^{-(n-1)}\sqrt{\det(g(Y_i(t), Y_j(t)))}}_{=: J(u,t)},$$

and we can rewrite the integral in (2) as

$$\mathrm{vol}(M, g) = \int_{S^{n-1}} \int_0^{\rho(u)} J(u,t) t^{n-1}\, dt du. \tag{3}$$

Note in particular that $J(u,t)$ is independent of our choice of $e_2, \ldots, e_n$, since also the integrand in (2) is.

Example 2.5. Consider S^n with its canonical measure. With the notation used above we have $Y_i(t) = \sin(t)E_i(t)$, where E_i is the parallel vector field satisfying $E_i(0) = e_i$. We get

$$\mathrm{vol}(S^n, \mathrm{can}) = \int_{S^{n-1}} \int_0^{\pi} \left(\frac{\sin t}{t}\right)^{n-1} t^{n-1}\, dt du = \mathrm{vol}(S^{n-1}, \mathrm{can}) \int_0^{\pi} (\sin t)^{n-1}\, dt,$$

from which the known formulas

$$\mathrm{vol}(S^{2n}, \mathrm{can}) = \frac{(4\pi)^n (n-1)!}{(2n-1)!} \quad \text{and} \quad \mathrm{vol}(S^{2n+1}, \mathrm{can}) = 2\frac{\pi^{n+1}}{n!}$$

can be recovered.

3 Volume Estimates

Let $V^k(r)$ denote the volume of a ball of radius r is the complete simply connected Riemannian manifold with constant curvature k. Recall that, for a manifold with constant sectional curvature a we have $\mathrm{Ric} = (n-1)ag$.

Theorem 3.1 (Bishop–Günther)**.** *Let (M, g) be a complete Riemannian manifold, and $B_p(r)$ be a ball which does not meet the cut-locus of p.*

(i) If there is a constant a such that $\mathrm{Ric} \geqslant (n-1)ag$*, then*

$$\mathrm{vol}(B_p(r)) \leqslant V^a(r).$$

(ii) If there is a constant b such that $K \leqslant b$, then

$$\mathrm{vol}(B_p(r)) \geqslant V^b(r).$$

Proof. We first proceed to establish the function $J(u,t)$ in terms of Jacobi fields that will be most convenient in our context. Let $u \in S^{n-1}$. Suppose that $0 \leqslant r \leqslant \rho(u)$.

For the chosen u, take the geodesic $c(t) = \exp_p tu$ from p and an orthonormal basis $\{u, e_2, \ldots, e_n\}$. For each $2 \leqslant i \leqslant n$, let us consider the parallel transport vector field E_i along c such that $E_i(0) = e_i$.

Note that for such a choice of r, $T_{ru} \exp_p : T_pM \to T_{c(r)}M$ is an isomorphism. Thus, there exists a unique $v_i \in T_pM$ such that $T_{ru} \exp_p(rv) = E_i(r)$. Then,

$$Y_i^r(t) := T_{tu} \exp_p(tv).$$

is a unique Jacobi field on c such that

$$\begin{aligned} Y_i^r(0) &= 0, \\ Y_i^r(r) &= E_i(r), \\ (Y_i^r)'(0) &= v_i. \end{aligned}$$

Now, using the above obtained expressions of Jacobi field, we obtain

$$J(u,t) = t^{1-n} \frac{\det\left(Y_2^r(t), \ldots, Y_n^r(t)\right)}{\det\left(Y'^r_2(0), \ldots, Y'^r_n(0)\right)} = t^{1-n} C_r \det\left(Y_2^r(t), \ldots, Y_n^r(t)\right).$$

Now, we set $f(t) = J(u,t)$. Now we introduce a new lemma that would let us compare the $f(t)$ of the chosen manifold to that of constant curvature a.

Before we proceed, let us recall some definitions and properties that will be used later.

1. For a differentiable map $A(t)$ from an interval in $\mathbb{R}$ to $GL_n(\mathbb{R})$, we have

$$(\det A)' = (\det A)\ \operatorname{tr}(A^{-1}A').$$

2. Note that, for a Jacobi field Y on c, we can consider its corresponding geodesic variation c_t. For this geodesic variation, we can consider its energy $\frac{1}{2}\int_0^r |c_t'(s)|^2 ds$ whose the second variation formula is given by

$$I(Y,Y) = \frac{d^2}{dt^2} E(c_t)|_{t=0} = \int_0^r \left(|Y'|^2 - R(Y, c', Y, c')\right) ds.$$

3. For the inner product of vector fields, we have the following identity:

$$g(X', Y') = (g(X, Y'))' - g(X, Y'').$$

Lemma 3.2. *Denoting I by the index form of energy, we have*

$$\frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I(Y_i^r, Y_i^r) - \frac{(n-1)}{r}.$$

Proof. From the definition of the Jacobi fields, we have

$$\left|\det\left(Y_2^r,\dots,Y_n^r\right)\right| = \left(\det g\left(Y_i^r,Y_j^r\right)\right)^{1/2}.$$

Let $D(t)$ denote the determinant $\det\left(g\left(Y_i^r,Y_j^r\right)\right)$. Then, we have

$$\begin{aligned} f'(t) &= C_r\left(-(n-1)t^{-n+2}(D(t))^{1/2} + t^{-(n-1)}\frac{D'(t)}{2(D(t))^{1/2}}\right) \\ \implies \frac{f'(t)}{f(t)} &= \frac{D'(t)}{2D(t)} - \frac{n-1}{t}. \end{aligned}$$

Note that, due the orthonormality of the parallel vector fields, for $t = r$, the matrix $\left[g(Y_i^r,Y_j^r)\right]$ is the unit matrix, from property 1, we have

$$D'(r) = 2\sum_{i=2}^{n} g\left((Y_i^r)',Y_i^r\right).$$

Now, using property 2 and 3, for each of the Jacobi fields Y_i^r, we obtain

$$\begin{aligned} I(Y,Y) &= \int_0^r \left(|Y'|^2 - R(Y,c',Y,c')\right) ds \\ &= \int_0^r \left((g(Y',Y))' - g(Y'',Y) + R(Y,c',c',Y)\right) ds \\ &= \int_0^r \left((g(Y',Y))' - g(R(Y,c')c',Y) + R(y,c',c',y)\right) \\ &= \int_0^r \left((g(Y',Y))' - R(Y,c',c',Y) + R(Y,c',c',Y)\right) = [g(Y,Y')]_0^r, \end{aligned}$$

and this proves our lemma. □

The following result was presented in Talk 3, on the Rauch–Jacobi field comparison theorem.

Lemma 3.3. *If $c : [a,b] \to M$ is a minimizing geodesic, Y is a Jacobi field and X is a vector field along c with the same values as Y at the ends, then $I(X,X) \geqslant I(Y,Y)$, with equality only if $X = Y$.*

Proof of (i). For each i, let us define a new vector field X_i^r on c give by,

$$X_i^r(t) = \frac{s(t)}{s(r)}E_i(t),$$

where

$$s(t) = \sin(\sqrt{a}t), \qquad \text{if } a > 0,$$

$$
\begin{aligned}
s(t) &= t, && \text{if } a = 0,\\
s(t) &= \sinh(\sqrt{-a}t) && \text{if } a < 0.
\end{aligned}
$$

From Lemma 3.3, we have

$$\sum_{i=2}^{n} I(Y_i^r, Y_i^r) \leqslant \sum_{i=2}^{n} I(X_i^r, X_i^r).$$

Now, each of the index form in the right satisfies

$$
\begin{aligned}
I(X_i^r, X_i^r) &= \int_0^r \left(g((X_i^r)', (X_i^r)') - R(X_i^r, c', X_i^r, c')\right) ds\\
&= \int_0^r \left((g(X_i^r, (X_i^r)'))' - g(X_i^r, (X_i^r)'') - R(X_i^r, c', X_i^r, c')\right) ds\\
&= \int_0^r \left(\left(\frac{s(t)}{s(r)}\right)^2 \left(a - R(E_i^r, c', E_i^r, c')\right)\right) ds + g(X_i^r, (X_i^r)')(r)\\
\implies \sum_{i=2}^{n} I(X_i^r, X_i^r) &= \int_0^r \left(\left(\frac{s(t)}{s(r)}\right)^2 ((n-1)a - \operatorname{Ric}(c', c'))\right) ds + \sum_{i=2}^{n} g\left(X_i^r, (X_i^r)'\right)(r).
\end{aligned}
$$

Note that, from the hypothesis of the theorem the integral in the above equation is negative. Further, using Lemma 3.2 and the definition of X_i^r, we obtain the following.

$$
\begin{aligned}
\frac{f'(r)}{f(r)} &\leqslant (n-1)\left(\sqrt{a}\operatorname{cotan}\sqrt{a}r - \frac{1}{r}\right) && \text{if } a > 0,\\
\frac{f'(r)}{f(r)} &\leqslant 0 && \text{if } a = 0,\\
\frac{f'(r)}{f(r)} &\leqslant (n-1)\left(\sqrt{-a}\operatorname{cotanh}\sqrt{-a}r - \frac{1}{r}\right) && \text{if } a < 0.
\end{aligned}
$$

In any of the above cases, if $f_a(r)$ denotes the function $J(u, r)$ for the space with constant curvature a, we have

$$\frac{f'(r)}{f(r)} \leqslant \frac{f_a'(r)}{f_a(r)}.$$

By integrating the above we obtain, $f(r) \leqslant f_a(r)$ and the result follows from further integration using the formula for volume.

Proof of (ii). Let Y denote one of the Jacobi fields (Y_i^r) defined earlier. Then, from the reduced equation for the index form, we have the following:

$$
\begin{aligned}
g(Y, Y')(r) &= \int_0^r (g(Y', Y') - R(Y, c', Y, c'))\, ds\\
&\geqslant \int_0^r (g(Y', Y') - bg(Y, Y))\, ds
\end{aligned}
$$

We can express the Jacobi fields, in terms of the parallel transports $E_i(t)$, as:

$$Y(t) = \sum_{i=2}^{n} y^i(t) E_i(t)$$

On the simply connected manifold with constant curvature b, take a geodesic manifold $\tilde{c}$ of length r, and define vector fields $\tilde{E}_i$ along $\tilde{c}$ in the same was as the vectors E_i. Set

$$\tilde{Y}(t) = \sum_{i=2}^{n} y^i(t) \tilde{E}_i(t).$$

Then we have

$$\int_0^r \left(|\tilde{Y}'|^2 - b|\tilde{Y}|^2 \right) dt = \int_0^r \left(|Y'|^2 - b|Y|^2 \right) dt = I(\tilde{Y}, \tilde{Y}).$$

Now, we apply Lemma 3.3 to the simply connected manifold with constant curvature b, and get

$$I(\tilde{Y}_i^r, \tilde{Y}_i^r) \geqslant I(\tilde{X}_i^r, \tilde{X}_i^r),$$

where $\tilde{X}_i^r(t) = \frac{s(t)}{s(r)} \tilde{E}_i(t)$ is the Jacobi field which takes at the ends of $\tilde{c}$ the same values as $\tilde{Y}_i^r$. Finally, using Lemma 3.2, we have

$$\frac{f'(r)}{f(r)} \geqslant \frac{f_b'(r)}{f_b(r)},$$

and the proof follows from integrating the above. □

The first estimate was further strengthened by Bishop and Gromov to obtain that the ratio $r \mapsto \frac{\mathrm{vol}(B_p(r))}{V^a(r)}$ is a nonincreasing function, whose limit is 1 as $r \to 0$.

Exercises

1. Let $p \in \mathbb{RP}^n$. Show that Cut_p is a submanifold isometric to $\mathbb{RP}^{n-1}$.
2. Prove Lemma 2.4.
3. Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geqslant (n-1)ag$ for some constant a, and assume that there exists a point $p \in M$ and radius $r > 0$ (small enough so that $B_p(r)$ does not meet Cut_p) with $\mathrm{vol}(B_p(r)) = V^a(r)$. The goal of this exercise is to prove that $B_p(r)$ is isometric to the corresponding ball in the model space of constant curvature a.
   (a) Prove that every Jacobi field J along a unit-speed geodesic c starting from p, satisfying $J(0) = 0$, $|J(r)| = 1$ and $\langle J, c' \rangle = 0$, is of the form

$$J(t) = \frac{s(t)}{s(r)} E_i(t) \quad \text{for } t \in [0, r],$$

with E_i and $s(t)$ as in the proof of Theorem 3.1.

Hint. Use the equality case of Lemma 3.3.

(b) Let $\tilde{p}$ be a point in the model space $\tilde{M}$ ($= S^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$) of constant curvature a, and $\iota : T_p M \to T_{\tilde{p}} \tilde{M}$ a linear isometry. Prove that

$$\exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1} |_{B_p(r)} : B_p(r) \to \tilde{M}$$

is an isometry onto its image.

Hint. Look at the proof of Cartan's theorem (Talk 2).

References

[1] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.