The Laplacian on Riem. mfds

<u>Def</u>: For $f \in C^\infty(\mathbb{R}^n)$, the Laplacian is defined as
$$\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{(\partial x^i)^2}$$

<u>Rem</u> The reason for this choice of sign convention will be clear later

<u>Def</u> For $f \in C^\infty(\mathbb{R}^n)$, we define

- The differential of f as $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \in \Omega^1(\mathbb{R}^n)$
- The gradient of f as $\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$
- The Hessian of f as $\operatorname{Hess} f = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j \in \mathcal{T}^0_2(\mathbb{R}^n)$

For $X = X^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$, we define $\operatorname{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i} \in C^\infty(\mathbb{R}^n)$

For $\omega = \omega_j dx^j \in \Omega^1(\mathbb{R}^n)$, we define $d^* \omega = -\sum_{j=1}^n \frac{\partial \omega_j}{\partial x^j} \in \Omega^0(\mathbb{R}^n)$

<u>Rem</u> ∘) With these definitions, we can introduce the Laplacian in different ways:
$$\Delta f = -\operatorname{div}(\operatorname{grad} f) = d^* d f = -\operatorname{tr}(\operatorname{Hess} f)$$

∘) If $f \in C^\infty_c(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \Delta f \, dx^1 \wedge \dots \wedge dx^n = -\int_{\mathbb{R}^n} \operatorname{div}(\operatorname{grad} f) dx^1 \wedge \dots \wedge dx^n = 0$

∘) A computation shows $\operatorname{div}(\varphi \cdot X) = \langle \operatorname{grad} \varphi, X \rangle + \varphi \cdot \operatorname{div} X$
$$\Rightarrow \operatorname{div}(\varphi \cdot \operatorname{grad} f) = \langle \operatorname{grad} \varphi, \operatorname{grad} f \rangle - \varphi \cdot \Delta f$$
$$\Rightarrow \int_{\mathbb{R}^n} \varphi \cdot \Delta f \, dx^1 \wedge \dots \wedge dx^n = \int_{\mathbb{R}^n} \langle \operatorname{grad} \varphi, \operatorname{grad} f \rangle dx^1 \wedge \dots \wedge dx^n$$

$= d^* d\varphi$, justifying the notation d^*

$$= \int_{\mathbb{R}^n} \langle d\varphi, df \rangle dx^1 \wedge \dots \wedge dx^n = \int_{\mathbb{R}^n} \Delta \varphi \cdot f \, dx^1 \wedge \dots \wedge dx^n$$
$$\forall f \in C^\infty(\mathbb{R}^n) \ \forall \varphi \in C^\infty_c(\mathbb{R}^n)$$

Def Let (M,g) be a Riem. mfd and $f\in C^\infty(M)$.

·) Recall that for $p\in M$ $T_pf: T_pM \to \mathbb{R}$ is linear

$\rightsquigarrow df(p)\in T_p^*M \implies df: p\mapsto df(p)$ $df\in\Omega^1(M)$

Locally, $df=\sum_{i=1}^n \frac{\partial f}{\partial x^i}dx^i = \partial_i f\, dx^i$ (Einstein summation convention)

df is called differential of f

·) $\operatorname{grad} f := (df)^\# \in \mathfrak{X}(M)$ i.e. $\langle \operatorname{grad} f, X\rangle = df(X)$ $\forall X\in\mathfrak{X}(M)$

Locally, $\operatorname{grad} f = g^{ij}\partial_j f \frac{\partial}{\partial x^i}$

$\operatorname{grad} f$ is called gradient of f

·) Let $X\in\mathfrak{X}(M)$, then we define $\operatorname{div} X = \operatorname{tr}(Y\mapsto \nabla_Y X)$ as the divergence of X

Locally, $\nabla_Y X = (Y^j\partial_j X^k + Y^j X^i \Gamma_{ji}^k)\partial_k \in \mathfrak{X}(M) = \mathcal{T}_0^1(M)$

$(Y\mapsto\nabla_Y X) = (\partial_j X^k + X^i\Gamma_{ji}^k)\, dx^j\otimes\partial_k \in \mathcal{T}_1^1(M)$

$\operatorname{tr}(Y\mapsto\nabla_Y X) = \partial_j X^j + X^i \Gamma_{ki}^k \in C^\infty(M)$

·) For $\omega\in\Omega^1(M) = \mathcal{T}_1^0(M)$, $\nabla\omega\in\mathcal{T}_2^0(M)$ is def. by

$$\nabla\omega(X,Y) := \nabla_Y\omega(X) = Y(\omega(X)) - \omega(\nabla_Y X)$$

and $d^*\omega\in C^\infty(M)$ is def by $-\operatorname{tr}_g\nabla\omega$

Locally, $\nabla\omega = \nabla\omega(\partial_i,\partial_j)\,dx^i\otimes dx^j$, with

$$\nabla\omega(\partial_i,\partial_j) = \partial_i(\omega(\partial_j)) - \omega(\nabla_{\partial_i}\partial_j)$$

$$= \partial_i\omega_j - \omega(\Gamma_{ij}^k\partial_k) = \partial_i\omega_j - \Gamma_{ij}^k\omega_k$$

·) The Laplace operator of (M,g) is

$$\Delta_g f = -\operatorname{div}(\operatorname{grad} f) \underset{\text{def}}{=} d^*df = -\operatorname{tr}_g(\nabla df) \underset{\text{def}}{=} -\operatorname{tr}_g(\nabla^2 f)$$

Locally, $\Delta f = -g^{ij}(\partial^2_{ij} f - \Gamma_{ij}^k\partial_k f)$

Rem $\operatorname{div} X = \operatorname{tr}(Y\mapsto\nabla_Y X) = \sum_{i=1}^n \langle\nabla_{e_i}X, e_i\rangle$, where $\{e_i\}$ is a local ONB

If $X=\operatorname{grad} f$, then $\operatorname{div}\operatorname{grad} f = \sum_{i=1}^n\langle\nabla_{e_i}\operatorname{grad} f, e_i\rangle \underset{[OG1]}{=} \sum_{i=1}^n \nabla^2_{e_i,e_i} f = \operatorname{tr}_g\nabla^2 f$

$\Rightarrow$ justifies the above equalities

Thm (Gauß divergence thm) If (M,g) is a Riem mfd and $X \in \mathfrak{X}(M)$ is a vector field with cpt. support, then $\int_M \operatorname{div} X \, dvol_g = 0$

$$\operatorname{div}(\varphi X) = \sum_{i=1}^n \langle \nabla_{e_i}(\varphi X), e_i \rangle = \sum_{i=1}^n \langle e_i(\varphi) \cdot X + \varphi \nabla_{e_i} X, e_i \rangle$$

$$= \langle X, \underbrace{\sum_{i=1}^n e_i(\varphi) \cdot e_i}_{= \langle \operatorname{grad}\varphi, e_i \rangle} \rangle + \varphi \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle$$

$$= \langle X, \operatorname{grad}\varphi \rangle + \varphi \operatorname{div} X$$

$$\Longrightarrow \int_M \varphi \cdot \Delta_g f \cdot dV_g = \int_M \langle \operatorname{grad}\varphi, \operatorname{grad} f \rangle dV_g = \int_M \Delta_g \varphi \cdot f \, dV_g$$

$$= \int_M \langle d\varphi, df \rangle dV_g \qquad \forall f \in C^\infty(M), \varphi \in C_c^\infty(M)$$

$= d^* d\varphi$, again justifying the notation

Rem .) The scalar product of $\omega, \eta \in \Omega^1(M)$ is def. by $\langle \omega, \eta \rangle := \langle \omega^\sharp, \eta^\sharp \rangle$

Locally, $\langle \omega, \eta \rangle = g^{ij} \omega_i \eta_j$

.) Locally, $dV_g = \sqrt{g} \, dx^1 \wedge \dots \wedge dx^n$, with $\sqrt{g} := \det(g_{ij})$

For $f, \varphi \in C^\infty(M)$, supported in a coord. nbhd, we get

$$\int_M \Delta f \cdot \varphi \, dV_g = \int_M \langle df, d\varphi \rangle dV_g = \int_M g^{ij} \partial_i f \, \partial_j \varphi \sqrt{g} \, dx^1 \wedge \dots \wedge dx^n$$

$$= -\int_M \partial_j(\sqrt{g} g^{ij} \partial_i f) \cdot \varphi \, dx^1 \wedge \dots \wedge dx^n$$

$$= -\int_M \sqrt{g}^{-1} \partial_j(\sqrt{g} g^{ij} \partial_i f) \cdot \varphi \underbrace{\sqrt{g} \, dx^1 \wedge \dots \wedge dx^n}_{dV_g}$$

$$\Longrightarrow \Delta f = -\frac{1}{\sqrt{g}} \partial_j(\sqrt{g} g^{ij} \partial_i f)$$

Def A function f on a Riem mfd (M,g) is called harmonic if $\Delta_g f = 0$

Rem The Laplace equation $\Delta_g f = 0$ is the Euler-Lagrange eq. of the functional $f \mapsto E(f) := \frac{1}{2} \int_M |df|_g^2 dV_g$:

For $\varphi \in C^\infty(M)$ with compact support,

$$\frac{d}{dt}\Big|_{t=0} E(f + t\varphi) = \int_M \langle df, d\varphi \rangle dV = \int_M \Delta_g f \cdot \varphi \, dV_g$$

The spectrum of the Laplacian on (S^n, g_{rd})

__Recall__: We define $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $g_{rd} := g_{eucl}|_{S^n}$

We have $(\mathbb{R}^{n+1} \setminus 0, g_{eucl}) \overset{isom}{\cong} ((0,\infty) \times S^n, dr^2 + r^2 g_{rd})$

__Prop__ Let $\Delta_{\mathbb{R}^{n+1}}, \Delta_{S^n}$ be the Laplacians of $(\mathbb{R}^{n+1}, g_{eucl}), (S^n, g_{rd})$, resp.
Then we have $\Delta_{\mathbb{R}^{n+1}} = -\partial^2_{rr} - \frac{n}{r}\partial_r + \frac{1}{r^2}\Delta_{S^n}$

__Proof__ Let $(U, \varphi = (x^1, \dots, x^n))$ be an arbitrary chart on S^n and $((0,\infty) \times U, id \times \varphi)$ the corresponding coord. of $\mathbb{R}^{n+1} \setminus \{0\} \cong (0,\infty) \times S^n$. Denote the r-coord. by 0.

Let $\{g_{ij}\}_{1 \le i,j \le n}$ the rep. of g_{S^n} in (U, φ), $\{\tilde{g}_{\alpha\beta}\}_{0 \le \alpha,\beta \le n}$ the rep of g_{eucl}

Then, $\tilde{g}_{00} = 1$, $\tilde{g}_{0i} = \tilde{g}_{i0} = 0$ for $1 \le i \le n$, $\tilde{g}_{ij} = r^2 g_{ij}$ for $1 \le i,j \le n$

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{k\alpha}(\partial_i \tilde{g}_{\alpha j} + \partial_j \tilde{g}_{\alpha i} - \partial_\alpha \tilde{g}_{ij}) \qquad (\alpha = 0, 1, \dots, n) \quad i,j,k \in \{1, \dots, n\}$$

No off-diag comp

$$= \frac{1}{2}\tilde{g}^{kl}(\partial_i \tilde{g}_{lj} + \partial_j \tilde{g}_{lj} - \partial_l \tilde{g}_{ij}) \qquad (l = 1 \dots n)$$

$$= \frac{1}{2}g^{ij}(\partial_i g_{lj} + \partial_j g_{lj} - \partial_l g_{ij}) \qquad (\tilde{g}^{kl} = r^{-2} g_{kl},\ \tilde{g}_{ij} = r^2 g_{ij})$$

$$= \Gamma^k_{ij}$$

$$\tilde{\Gamma}^0_{ij} = \frac{1}{2}\tilde{g}^{0\alpha}(\partial_i \tilde{g}_{\alpha j} + \partial_j \tilde{g}_{\alpha i} - \partial_\alpha \tilde{g}_{ij}) \qquad \tilde{g}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (r^2 g)^{-1} \end{pmatrix}$$

No off diag comp.

$$= \frac{1}{2}(\partial_i \underbrace{\tilde{g}_{0j}}_{=0} + \partial_j \underbrace{\tilde{g}_{0i}}_{=0} - \partial_0 \underbrace{\tilde{g}_{ij}}_{r^2 g_{ij}}) = -r g_{ij}$$

$$\tilde{\Gamma}^\alpha_{00} = \frac{1}{2}\tilde{g}^{\alpha\beta}(\partial_0 \underbrace{\tilde{g}_{\beta 0}}_{const} + \partial_0 \underbrace{\tilde{g}_{\beta 0}}_{const} - \partial_\beta \underbrace{\tilde{g}_{00}}_{const}) = 0$$

$$\Rightarrow \Delta_{\mathbb{R}^{n+1}} = -\tilde{g}^{\alpha\beta}(\partial^2_{\alpha\beta} - \tilde{\Gamma}^\gamma_{\alpha\beta}\partial_\gamma)$$

$$= -\underbrace{\tilde{g}^{00}}_{=1}(\partial^2_{00} - \underbrace{\tilde{\Gamma}^\gamma_{00}}_{=0}\partial_\gamma) - \underbrace{\tilde{g}^{ij}}_{=r^{-2}g^{ij}}(\partial^2_{ij} - \underbrace{\tilde{\Gamma}^0_{ij}}_{-r g_{ij}}\partial_0 - \underbrace{\tilde{\Gamma}^k_{ij}}_{=\Gamma^k_{ij}}\partial_k)$$

$$= -\partial^2_{00} - r^{-2}g^{ij}(\partial^2_{ij} - \Gamma^k_{ij}\partial_k) - r^{-1}\underbrace{g^{ij}g_{ij}}_{\delta^i_i = n}\partial_0$$

$$= -\partial^2_{00} - r^{-1} \cdot n\,\partial_0 + r^{-2}\Delta_{S^n}$$

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Let $H \in C^\infty(\mathbb{R}^{n+1})$ be a homogeneous polynomial of degree k.
Then $H(x) = \|x\|^k H\left(\frac{x}{\|x\|}\right) \; \forall x \in \mathbb{R}^{n+1} \setminus \{0\}$. In other words, it can be written as $H(x) = r^k \Phi(y)$ on $\mathbb{R}^{n+1} \setminus \{0\} \cong (0,\infty) \times S^n$ for some $\Phi \in C^\infty(S^n)$
If H is harmonic, we obtain

$$0 = \Delta_{\mathbb{R}^{n+1}} H = \left(-\partial_{rr}^2 - \frac{n}{r}\partial_r + \frac{1}{r^2}\Delta_{S^n}\right)(r^k \Phi)$$
$$= -k(k-1) r^{k-2} \Phi - n \cdot k\, r^{k-2}\Phi + r^{k-2}\Delta_{S^n}\Phi$$
$$= -k(n+k-1)\, r^{k-2}\Phi + r^{k-2}\Delta_{S^n}\Phi$$

$\Rightarrow \Delta_{S^n}\Phi = k(n+k-1)\Phi$, i.e. Φ is an eigenfunction of Δ_{S^n}, with eigenvalue $k(n+k-1)$

For example, if H is of degree $k=0$, it is constant, hence harmonic
$\Rightarrow \Phi$ will also be constant on S^n and $\Delta_{S^n}\Phi = 0$
If H is hom. of degree 1, i.e. $H(x) = \sum_{i=1}^{n+1} a_i x^i$, $a_i \in \mathbb{R}$, then $\Delta_{\mathbb{R}^{n+1}} H = 0 \Rightarrow$ harmonic
$\Rightarrow \Delta_{S^n}\Phi = (n-1)\Phi$

Let $P_k = \{H \in C^\infty(\mathbb{R}^{n+1}) \mid H \text{ hom. pol. of deg. } k\}$,
$H_k = \{H \in P_k \mid \Delta_{\mathbb{R}^{n+1}} H = 0\}$
We have seen $P_0 = H_0$, $P_1 = H_1$. One can show $P_k = H_k \oplus r^2 P_{k-2} \; \forall k \geq 2$

$$\Rightarrow \quad P_{2k} = H_{2k} \oplus r^2 H_{2k-2} \oplus \dots \oplus r^{2k} H_0$$
$$P_{2k+1} = H_{2k+1} \oplus r^2 H_{2k-1} \oplus \dots \oplus r^{2k} H_1$$

In particular, all H_k are nonempty and all $k(n+k-1)$ are eigenvalues of Δ_{S^n} with corresponding eigenspaces $E_{k(n+k-1)} = \{H|_{S^n} \in C^\infty(S^n) \mid H \in H_k\}$
Stone-Weierstrass theorem: Let $K \subset \mathbb{R}^{n+1}$ be a cpt. subset. Then for every $f \in C^\infty(\mathbb{R}^{n+1})$, there exists a sequence (f_ℓ) of polynomials s.t. $\|f_\ell - f\|_{C^0(K)} \xrightarrow{\ell\to\infty} 0$
$\Rightarrow \bigoplus_{k\geq 0} E_{k(n+k-1)}$ is C^0 dense in $C^\infty(S^n)$, hence also L^2-dense in $C^\infty(S^n)$, and $C^\infty(S^n)$ is L^2-dense in $L^2(S^n)$
$\Rightarrow \bigoplus_{k\geq 0} E_{k(n+k-1)}$ is L^2 dense in $L^2(S^n)$
With the results of next week, we can say that we found all eigenvalues and eigenspaces of Δ_{S^n}