THE SPECTRUM OF THE LAPLACIAN ON RIEMANNIAN MANIFOLDS

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1. The spectrum of the Laplacian

This section of these lecture notes are based in its entirety on Chapter 3.2 in Riemannian Geometry and Geometric Analysis by Jürgen Jost [2]

In order to make the statements of these sections precise, we will need a few results from functional analysis, specifically from the analysis on Sobolev Spaces. Let us first define the Sobolev space we work on. M will always be a compact Riemannian manifold. For $f, h \in C^\infty(M)$, we define

$$(f, h) = \int_M f(x)h(x)\sqrt{g}dx_1 \cdots dx_d,$$

and

$$\begin{aligned}(df, dh) &= \int_M \langle df, dh\rangle \sqrt{g}dx_1 \cdots dx_d \\ &= \int_M g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}\sqrt{g}dx_1 \cdots dx_d.\end{aligned}$$

Here and throughout $\sqrt{g} = \sqrt{\det g_{ij}}$ using the local coordinates x_i. Further, we use the notation

$$\|f\|_{L^2} = \sqrt{(f, f)}, \quad \|df\|_{L^2} = \sqrt{(df, df)}$$

and $L^2(M)$ is the closure of $C^\infty(M)$ using the $\|\cdot\|_{L^2}$-norm. Next,

$$\langle f, h\rangle = (f, h) + (df, dh).$$

Let $\|f\|_{W^{1,2}} = \sqrt{\langle f, f\rangle}$ and define $H^{1,2}$ as the closure of $C^\infty(M)$ using the $\|\cdot\|_{W^{1,2}}$-norm. To simplify notation we will write $H = H^{1,2}$.

Our goal is to analyze the functions $f \in H$ and the values λ such that

$$\Delta f = \lambda f. \tag{1.1}$$

From the previous lecture we know that

$$(\Delta f, h) = (df, dh) = (f, \Delta h)$$

for all smooth f and h (recall that M is compact) so if $\Delta f = \lambda f$,

$$\lambda(f, f) = (\Delta f, f) = (df, df)$$

implies that $\lambda \geq 0$.

We may note that if $f \equiv c$, c constant, then $\lambda = 0$ is a solution and shall later see that this is the only situation when $\lambda = 0$ occurs. In order to further analyze the situation, we state, without proof, results from functional analysis that will be needed. Many of these statement may be generalized to a wider class of function spaces, but we state them in the version we need them here. For a somewhat more general setting, see Appendix A in [2].

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Theorem 1.1. *If $f \in H$ and $\int_M f = 0$, then*

$$\|f\|_{L^2} \leq C\|df\|_{L^2}$$

where C is a uniform constant independent of f.

Theorem 1.2. *H is compactly embedded in $L^2(M)$, i.e. if $(f_n) \subset H$ is uniformly bounded in the $\|\|_{W^{1,2}}$-norm, then a subsequence converges in L^2.*

We say that the sequence f_n converges weakly to f in H if

$$\langle f_n, w\rangle \to \langle f, w\rangle$$

for all $w \in H$.

Theorem 1.3. *If H_0 is a Hilbert space with norm $\|\cdot\|$, then every uniformly bounded sequence (f_n) in H_0 contains a weakly convergent subsequence and if the limit is f, then*

$$\|f\| \leq \liminf \|f_n\|$$

Remark 1.4. If (f_n) and (df_n) are uniformly bounded, then after extraction of a subsequence, f_n converges weakly in H to f and in $L^2(M)$ to f.

The last statement we will need is the following.

Theorem 1.5. *If $f \in H$ solves*

$$\int_M \langle df, d\phi\rangle \sqrt{g} dx_1 \cdots dx_d = \lambda \int_M f\phi\sqrt{g} dx_1 \cdots dx_d$$

for all $\phi \in H$, then $f \in C^\infty(M)$ and

$$\Delta f = \lambda f.$$

Let $v_0 = 1$ be the first (trivial) solution corresponding to the eigenvalue 0. We start by finding the first non-trivial solution to Eq. (1.1). We want to find

$$\lambda_1 = \inf_{f\in H\setminus\{0\}, \int_M f=0} \frac{(df, df)}{(f, f)}.$$

Theorem 1.1 yields $\lambda_1 > 0$. Take f_n a sequence such that

$$\lambda_1 = \lim_{n\to\infty} \frac{(df_n, df_n)}{(f_n, f_n)}.$$

By linearity we may assume that $\|f_n\|_{L^2} = 1$ for all n and by the definition of λ_1 that $\|df_n\|_{L^2} \leq K$ for all n. By Remark 1.4 after extraction of a subsequence, f_n converges weakly to $v_1 \in H$ and in L^2 to v_1. Since $\|f_n\|_{L^2} = 1$, this implies $\|v_1\|_{L^2} = 1$. Then using Theorem 1.3 and the fact that $\|df\|_{L^2}$ defines a norm in H,

$$\lambda_1 \leq (dv_1, dv_1) \leq \liminf_{n\to\infty}(df_n, df_n) = \lim_{n\to\infty}(df_n, df_n) = \lambda_1$$

so we in fact have equality in all steps.

We shall now make an inductive construction. Let us assume that (λ_i, v_i) has been constructed as above for $i = 0, ..., m-1$ and $\lambda_i \leq \lambda_{i+1}$, $\Delta v_i = \lambda_i v_i$ and finally that

$$(v_i, v_j) = \delta_{ij}.$$

We define

$$H_m = \{f \in H : (f, v_i) = 0 : i = 0, \cdots m-1\}$$

i.e. the orthogonal complement to the span of the v_i, $i = 0, \cdots m-1$. Note that if $(f_n) \subset H_m$ converges to f, then $(f_n, v_i) = 0$ for $i = 0, ..., m-1$ so $(f, v_i) = 0$ so $f \in H_m$. Hence H_m is closed and being the orthogonal complement of a finite dimensional subspace, it is also a Hilbert space for all m.

Then set

$$\lambda_m = \inf_{f \in H_m} \frac{(df, df)}{(f, f)}$$

and since $H_j \supset H_{j+1}, \lambda_{j+1} \geq \lambda_j$.

As above we find $v_m \in H_m$ with $\|v_m\|_{L^2} = 1$ such that

$$\lambda_m = (dv_m, dv_m) = \frac{(dv_m, dv_m)}{(v_m . v_m)}.$$

Now, take $\phi \in H_m$ and $t \in \mathbb{R}$. By definition of λ_m and v_m,

$$\lambda_m \leq \frac{(d(v_m + t\phi), d(v_m + t\phi))}{(v_m + t\phi, v_m + t\phi)}$$

and the right hand side is differentiable with respect to t and has a minimum in $t = 0$. Differentiating formally yields

$$0 = 2 \left(\frac{(dv_m, d\phi)}{(v_m, v_m)} - \frac{(dv_m, dv_m)}{(v_m, v_m)} \frac{(v_m, \phi)}{(v_m, v_m)} \right).$$

Since $(dv_m, dv_m) = \lambda_m$ and $(v_m, v_m) = 1$, we find

$$0 = 2((dv_m, d\phi) - \lambda_m (v_m, \phi))$$

and this holds for all $\phi \in H_m$. Take now instead $\phi \in H$, Since $(v_m, v_i) = 0$ for $i = 0 \cdots m - 1$ and $(dv_m, dv_i) = (dv_i, dv_m) = \lambda_i (v_i, v_m) = 0$, we find in fact that

$$(dv_m, d\phi) - \lambda_m (v_m, \phi)$$

for all $\phi \in H$. Hence

$$\int_M \langle dv_m, d\phi \rangle \sqrt{g} dx_1 \cdots dx_d = \lambda_m \int_M v_m \phi \sqrt{g} dx_1 \cdots dx_d$$

for all $\phi \in H$, and by Theorem 1.5, $v_m \in C^\infty(M)$ and

$$\Delta v_m = \lambda_m v_m.$$

Next, we see that $\lim_{m \to \infty} \lambda_m = \infty$. Indeed, if this was not the case, then a subsequence of (dv_m) would be uniformly bounded, hence have a convergent subsequence in L^2 by Theorem 1.2, i.e.

$\|v_{m_j} - v\|_{L^2} \to 0$. taking $j \neq l$, we find

$$\|v_{m_j} - v_{m_l}\|_{L^2}^2 = (v_{m_j}, v_{m_j}) - 2(v_{m_j}, v_{m_l}) + (v_{m_l}, v_{m_l}) = 2$$

contradicting that (v_{m_j}) is Cauchy.

We are ready to state the main theorem of this lecture, a large part of which we have already proven.

Theorem 1.6. *Let M be a compact Riemannian manifold.*

$$\Delta f = \lambda f, \quad f \in H^{1,2}$$

has a countable set of solutions (v_n, λ_n), i.e.

$$\Delta v_n = \lambda_n$$

for which

$$(v_m, v_n) = \delta_{nm},$$

$$(dv_n, dv_m) = \lambda_n \delta_{nm}$$

Furthermore, $\lambda_{n+1} \geq \lambda_n$, $\lambda_n = 0$ if and only if $n = 0$ corresponding to the constant functions and

$$\lim_{n \to \infty} \lambda_n = \infty.$$

For $f \in L^2(M)$,

$$f = \sum_{i=0}^{\infty} (f, v_i) v_i \tag{1.2}$$

and for $f \in H^{1,2}$,

$$(df, df) = \sum_{i=0}^{\infty} \lambda_i (f, v_i)^2. \tag{1.3}$$

Proof. Above we concluded that $\Delta v_n = \lambda_n v_n$. By construction of v_j we have

$$(v_m, v_n) = \delta_{nm},$$

and since

$$(dv_m, d\phi) - \lambda_m (v_m, \phi) = 0 \tag{1.4}$$

for all $\phi \in H$, it follows that

$$(dv_m, dv_n) = \lambda_n \delta_{nm}.$$

It remains to prove (1.2) and (1.3). We begin with the former. Write $(v_i, f) = a_i$. Denote by f_m the function

$$\sum_{i=0}^{m} a_i v_i$$

and $\phi_m = f - f_m$. We shall show that $\phi_m \to 0$ so that $f_m \to f$ in $L^2(M)$. To that end, we clearly have

$$(\phi_m, v_i) = 0 \tag{1.5}$$

for $i = 0, 1 \cdots m$ and so $\phi_m \in H_{m+1}$. Hence,

$$\lambda_{m+1} \le \frac{(d\phi_m, d\phi_m)}{(\phi_m, \phi_m)}.$$

We have from (1.4) and (1.5) that

$$(d\phi_m, dv_i) = 0, \quad i = 0, ..., m. \tag{1.6}$$

Moreover,

$$\begin{aligned}(\phi_m, \phi_m) &= (f - f_m, f - f_m) = (f, f) - 2(f, f_m) + (f_m, f_m) \\ &= (f, f) - 2(\phi_m + f_m, f_m) + (f_m, f_m) = (f, f) - (f_m, f_m)\end{aligned} \tag{1.7}$$

by (1.5). In the same vein but using (1.6) we conclude

$$(d\phi_m, d\phi_m) = (df, df) - (df_m, df_m). \tag{1.8}$$

We find that

$$(\phi_m, \phi_m) \le \frac{(d\phi_m, d\phi_m)}{\lambda_{m+1}} = \frac{(df, df) - (df_m, df_m)}{\lambda_{m+1}} \le \frac{(df, df)}{\lambda_{m+1}} \tag{1.9}$$

and since $\lambda_m \to \infty$, $\phi_m \to 0$ in $L^2(M)$. This implies

$$f = \lim_{m \to \infty} f_m = \sum_{i=0}^{\infty} a_i v_i$$

and (1.2) is proved.

Next,

$$df_m = \sum_{i=0}^{m} a_i dv_i.$$

Since

$$(dv_m, dv_n) = \lambda_n \delta_{nm},$$

$$(df_m, df_m) = \sum_{i=0}^{m} \lambda_i a_i^2.$$

By (1.8), $(df_m, df_m) \leq (df, df)$ for all m and all λ_i are non-negative. Hence

$$\sum_{i=0}^{m} \lambda_i a_i^2$$

is a monotone sequence bounded from above and thereby converges. If $n \geq m$, then using (1.6)

$$(d\phi_m - d\phi_n, d\phi_m - d\phi_n) = (df_n - df_m, df_n - df_m) = \sum_{i=m+1}^{n} \lambda_i a_i^2,$$

so $(d\phi_n)$ is Cauchy in $L^2(M)$. As we have already seen, $\phi_m \to 0$ in $L^2(M)$ so $\phi_m \to 0$ in $H^{1,2}$. (1.8) readily yields

$$(df, df) = \lim_{m \to \infty} (df_m, df_m) = \sum_{i=0}^{\infty} \lambda_i a_i^2$$

and (1.3) is proved. We will now verify that there are no other eigenvalues than the λ_j and that all eigenvectors are linear combinations of the v_j. To that end, suppose there are two eigenvalues $a \neq b$ such that

$$\Delta u = au, \quad \Delta v = bv.$$

Then for all $\phi \in H$,

$$(du, d\phi) = a(u, \phi)$$

$$(dv, d\phi) = b(v, \phi).$$

In particular, this is true when choosing $\phi = v$ and $\phi = u$ respectively. This yields

$$a(u, v) = (du, dv) = (dv, du) = b(v, u) = b(u, v)$$

so

$$(u, v) = 0.$$

Now, if there was an eigenvalue a not equal to any of the λ_m and a corresponds to an eigenvector v, then $(v, v_i) = 0$ for all i so that $v = 0$ by (1.2). However, $v = 0$ has the eigenvalue 0 and $\lambda_0 = 0$, contradicting that $a \neq \lambda_m$ for all m. Hence, the λ_j are all the eigenvalues. As $\lambda_j \to \infty$, each eigenvalue corresponds to only finitely v_j. Thereby, for any eigenvector

$$v = \sum_{i=0}^{\infty} (v, v_i) v_i$$

only finitely many terms are non-zero. This finishes the proof. □

2. The wave and heat equations

This section is very closely based on Section 4, Chapter 1 in Eigenvalues in Riemannian Geometry by Isaac Chavel[1]. Let us now see how we can use the spectral decomposition we found in the previous section to analyse the wave and heat equations on Riemannian manifolds. Notice that the following analysis is a sketch and a complete verification of the results would need more stringency. The situation below also covers the situation when M has a non-empty boundary. If this is not the case, you may simply consider the conditions on the boundary values as trivially true. Let us first analyze the wave equation. Then, we think of M as a membrane with fixed boundary and we would like to find the transverse vibration of this membrane. We are then looking for a function $v : M \times [0, \infty) \to \mathbb{R}$ (the second variable being time, t,) such that

$$\Delta v + \rho/\tau \frac{\partial^2 v}{\partial t^2}, \tag{2.1}$$
$$v(x, t) = 0$$

if $x \in \partial M$. Here, ρ is the density and τ the tension of the membrane. The method we utilize is one where we separate the two variables x and t. That is, we look for a solution

$$v(x, t) = X(x)T(t).$$

Putting this into (2.1) yields

$$\Delta X(x)T(t) + \rho/\tau X(x)T''(t).$$

Dividing with $X(x)T(t)$ gives

$$\frac{\Delta X(x)}{X(x)} = -\rho/\tau \frac{T''(t)}{T(t)}.$$

The left hand side is constant in x and the right hand side is constant in t. Hence the two sides must be constant and we obtain

$$T''(t) = -\frac{\lambda\tau}{\rho} T(t)$$
$$\Delta X = \lambda X$$

for some λ and $X(x) = 0$ if $x \in \partial M$. In the previous section we analyzed the solutions to the second equation. The former equation has solution

$$T(t) = A \cos(\sqrt{\lambda\tau/\rho}(t - B))$$

where A, B are arbitrary constants and $\lambda = \lambda_m$ for some eigenvalue λ_m of the Laplacian. By linearity of (2.1), the sums of solutions are again solutions, and we find that the possible solutions we can find using this separation technique are of the form

$$v(x, t) = \sum_{m=0}^{\infty} A_m v_m(x)(\cos(\sqrt{\lambda_m \tau/\rho}(t - B_m)))$$

where v_m are as in the previous section and A_m, B_m arbitrary constants.

If we are given initial conditions

$$v(x, 0) = f(x), \quad \frac{\partial v}{\partial t}(x, 0) = 0,$$

the latter implies that $B_m = 0$. Then

$$v(x, t) = \sum_{m=0}^{\infty} A_m v_m(x)(\cos(\sqrt{\lambda\tau/\rho} t))$$

and so

$$f(x) = \sum_{m=0}^{\infty} A_m v_m(x).$$

(1.2) now gives that $A_m = (f, v_m)$. Hence, we can get rid of the constants A_m in the expression for $v(x, t)$ by setting

$$w(x, y, t) = \sum_{m=0}^{\infty} v_m(y)v_m(x)(\cos(\sqrt{\lambda\tau/\rho} t))$$

and finding

$$v(x, t) = \int_M w(x, y, t) f(y) \sqrt{g} dy_1 \cdots dy_d.$$

Next, for the heat equation we look for a temperature function $v : M \times [0, \infty)$ solving (after a normalization of the physical constants)

$$\Delta v + \frac{\partial v}{\partial t} = 0, \tag{2.2}$$

such that $\nu_x v(x,t) = 0$ if $x \in \partial M$, i.e. that no heat leaves M (ν_x denotes the derivative in the normal direction at the boundary point $x \in \partial M$). We once again separate the two variables and posit $v(x,t) = X(x)T(t)$ and obtain

$$\Delta X = \lambda X(x), \quad T'(t) = -\lambda T(t)$$

and $\nu_x X(x) = 0$ on the boundary of M. The latter has solutions

$$T = A \exp(-\lambda t).$$

Supposing that $v(x,0) = f(x)$ we obtain as above that if

$$w(x,y,t) = \sum_{m=0}^{\infty} v_m(y) v_m(x) A \exp(-\lambda_m t)$$

we find

$$v(x,t) = \int_M w(x,y,t) f(y) \sqrt{g} dy_1 \cdots dy_d.$$

3. Exercises

Exercise 1. Find the eigenvalues and corresponding eigenfunctions of the Laplacian on the circle of radius R,

$$\mathbb{T}_R = \{x \in \mathbb{R}^2, |x| = R\}.$$

Exercise 2. Let $T^n = \mathbb{R}^n/\Gamma$ be a n-dimensional torus where

$$\Gamma = \{\sum_i^n a_i v_i; \quad a_i \in \mathbb{Z}, \quad (v_i) \text{ forms a basis of } \mathbb{R}^n\}$$

is a lattice. Consider the dual lattice $\Gamma*$, given by all $w^* \in \mathbb{R}^n$ such that $\langle w, w^* \rangle \in \mathbb{Z}$ for all $w \in \Gamma$ (where $\langle , \rangle$ is the Euclidian scalar product).

Show that every eigenfunction of the Laplacian is of the form

$$f(x) = \exp(2\pi i \langle w^*, x \rangle) \quad w^* \in \Gamma^*,$$

that any function of this form is an eigenfunction and that its corresponding eigenvalue is $4\pi^2 |w^*|^2$.

Hint: Use Fourier analysis to show that these are all eigenfunctions.

Exercise 3. Let M be a compact connected Riemannian manifold. Using separation of variables, find the solutions $\psi(x,t) : M \times \mathbb{R} \to \mathbb{C}$ of the normalized Schrödinger equation:

$$\Delta \psi(x,t) + i \frac{\partial}{\partial t} \psi(x,t) = 0.$$

References

[1] I. Chavel, B. Randol, and J. Dodziuk. *Eigenvalues in Riemannian Geometry.* ISSN. Elsevier Science, 1984. ISBN: 9780080874340. URL: https://books.google.se/books?id=0v1VfTWuKGgC.

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