The Laplace Operator on Differential Forms

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As we have seen, there is a natural way to extend the Laplace operator on functions in the Euclidean setting, to functions on Riemannian manifolds. In these notes, we will further extend the Laplace operator to now be defined on differential forms. The motivations for this extension is manifold (pun intended), but one such motivating fact is that, as we will see in the next talk, each cohomology class in $H^k(M)$ is uniquely represented by a harmonic k-form.

The text is primarily based on chapter 3.3 of [1].

1 Preliminaries

If the reader is unaware of the concept of the exterior product of vector spaces, please read chapter 7.1 of the Spring Differential Geometry notes. However, there is a slight difference. In the Spring notes, they define $\Lambda^p V$ to be what we will call $\Lambda^p(V^*)$ in this text. With this, everything you need to know, you will be able to read up in there. However, during the talk, it seemed like some of you may be unfamiliar with these concepts so we added the following paragraph for clarification.

Let V be a finite dimensional real vector space. Recall that since V is finite dimensional, we have that the tensor product $\bigotimes_{d=1}^{p} V$ is canonically isomorphic to the space of multi-linear maps $L : V^* \times \ldots \times V^* \to \mathbb{R}$ (it may even be defined in this way, as it is done in the spring DG-course). We of course have the analogous identification with $\bigotimes_{d=1}^{p} V^*$ and the space of all multilinear maps $L : V \times \ldots \times V \to \mathbb{R}$. With this, one way of defining the exterior product $\Lambda^p V$ would then be to define it as the subspace of $\bigotimes_{d=1}^{p} V$ for which all multilinear maps $L : V^* \times \ldots \times V^* \to \mathbb{R}$ satisfies $L(-, \ldots, -, v_i, -, \ldots, -, v_j, - \ldots, -) = -L(-, \ldots, -, v_j, -, \ldots, -, v_i, - \ldots, -)$.

On the other hand, if you want to view it in the classical sense of the tensor product you can define it as

$$\Lambda^p V = (\bigotimes_{d=1}^{p} V)/W,$$

where W is the subspace generated by the set $\{v_1 \otimes \ldots \otimes v_n \mid \exists i \neq j, \text{ for which } v_i = v_j\}$. The proof of lemma 7.2.1 in the spring DG notes clarifies why these two definitions are the same.

1.1 On $\mathbf{R}^n$

If V is a n-dimensional vector space with scalar product $\langle \cdot, \cdot \rangle$, we can construct a scalar product on $\Lambda^p V$, $1 \leq p \leq n$, by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle). \tag{1}$$

If $\{v_i\}$ is some orthonormal basis of V, then

$$v_{i_1} \wedge \cdots \wedge v_{i_p}, \ 1 \leq i_1 < \cdots < i_p \leq n, \tag{2}$$

is an orthonormal basis of $\Lambda^p V$.

Remark 1.1. The sorting of the indices i_k in Equation (2) results in easier book-keeping. We can 'sort' any such vector via

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{i_{k+1}} \wedge \cdots \wedge v_{i_p} = (-1) v_{i_1} \wedge \cdots \wedge v_{i_{k+1}} \wedge v_{i_k} \wedge \cdots \wedge v_{i_p}. \tag{3}$$

In particular, if some pair of indices is equal, $i_a = i_b$, then Equation (3) illustrates that the corresponding vector $v_{i_1} \wedge \cdots \wedge v_{i_p}$ is zero.

We can orient V by defining some basis $\{v_i\}$ as positive. Any other basis $\{Av_i\}$ is positive if $\det A > 0$, otherwise negative. From now on, assume that V is oriented.

Definition 1.1. Let V be an oriented vector space of dimension n. For each $0 \leq p \leq n$, we define the linear star operator

$$*_p : \Lambda^p(V) \to \Lambda^{n-p}(V)$$

via mapping each basis vector $e_{i_1} \wedge \ldots \wedge e_{i_p}$ to

$$\begin{cases} e_{j_1} \wedge \ldots \wedge e_{j_{n-p}} \text{ if } e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_{n-p}} \text{ is a positive basis for } V \\ -e_{j_1} \wedge \ldots \wedge e_{j_{n-p}} \text{ if } e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_{n-p}} \text{ is a negative basis for } V \end{cases}$$

and then extending linearly to all of $\Lambda^p(V)$.

The $*$-map is a vector space isomorphism. A particular property of the $*$-map is that for a orthonormal basis $\{v_i\}$

$$\begin{aligned} *(1) &= v_1 \wedge \cdots \wedge v_n \\ *(v_1 \wedge \cdots \wedge v_n) &= 1. \end{aligned} \tag{4}$$

Also, we can show that for some $n \times n$ matrix A

$$*(Av_{i_1} \wedge \cdots \wedge Av_{i_n}) = \det(A) * v_{i_1} \wedge \ldots \wedge v_{i_n}. \tag{5}$$

Lemma 1.1. *The map* $* \circ * : \Lambda^p(V) \to \Lambda^p(V)$ *equals* $(-1)^{p(n-p)} : \Lambda^p(V) \to \Lambda^p(V)$.

Proof. Exercise. □

Lemma 1.2. *For* $v, w \in \Lambda^p(V)$*, we have*

$$\langle v, w\rangle = *(w \wedge *v) = *(v \wedge *w) \tag{6}$$

Proof. We only need to show Equation (6) for the positive basis vectors $\{e_i = v_{i_1} \wedge \cdots \wedge v_{i_p}\}$. The general result then follows from the linearity of $\langle \cdot, \cdot \rangle$, and $*$.

First, let $e_1 \perp e_2$. Then there is some pair indices i_a, j_b in

$$e_1 = v_{i_1} \wedge \cdots \wedge v_{i_p},\ *e_2 = v_{j_1} \wedge \cdots \wedge v_{j_{n-p}} \tag{7}$$

such that $v_{i_a} = v_{j_b}$. Thus $e_1 \wedge *e_2 = 0$. If $e_1 = \pm e_2$, then

$$e_1 \wedge *e_2 = \pm v_1 \wedge \cdots \wedge v_n. \tag{8}$$

By Equation (4), $*(e_1 \wedge *e_2) = \pm 1$. □

Lemma 1.3. *Let* $\{v_i\}$ *be a positive basis of* V[1]*. Then*

$$*(1) = \frac{1}{\sqrt{\det\langle v_i, v_j\rangle}} v_1 \wedge \cdots \wedge v_n. \tag{9}$$

Proof. If $\{e_i\}$ is some orthonormal basis of V, then

$$v_1 \wedge \cdots \wedge v_d = \sqrt{\det\langle v_i, v_j\rangle} e_1 \wedge \cdots \wedge e_n. \tag{10}$$

The conclusion then follows from Equation (4). □

[1] Not necessarily orthonormal

1.2 On Riemannian Manifolds

Let (M, g) be a Riemannian manifold with orientation. We may then orient T_pM, and T_p^*M in a consistent manner.

Define the Euclidean basis $\{\frac{\partial}{\partial x_i} \in \mathbf{R}^n\}$ to be positive. A chart transition has positive determinant. Thus, we can define $\{d\phi^{-1}(\frac{\partial}{\partial x_i})\}$ to be positive.

Since $g^{ij} = (g_{ij})^{-1}$, by Lemma 1.3, we have

$$*(1) = \frac{1}{\sqrt{\det g^{ij}}} dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n. \tag{11}$$

This we can identify as the *volume form*

$$vol(M) = \int_M *(1). \tag{12}$$

With this we can define the L^2-product on (M, g):

Definition 1.2. Let α, $\beta \in \Omega^p(M)$. Define the L^2-product by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle * (1) = \int_M \alpha \wedge *\beta. \tag{13}$$

Further, define the L^2-norm as $|\alpha| := (\alpha, \alpha)^{1/2}$.

2 Laplace Operator on Forms

We follow the same approach as in [1] and assume that the manifold M is compact (and of course still also orientable). Only slight changes regarding some compact support assumptions must be made to fit the non-compact cases and the reader may see it as an exercise to figure out the details.

Definition 2.1. We define the operator d^* as the formal adjoint to d on $\bigoplus_{p=0}^n \Omega^p(M)$ with regards to $(\cdot, \cdot)$. That is, for all $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$, d^* is the operator satisfying

$$(d\alpha, \beta) = (\alpha, d^*\beta). \tag{14}$$

Thus, $d_p^* : \Omega^p(M) \to \Omega^{p-1}(M)$. However, we shall often omit the subscript and just write d to be a map from $\bigoplus_{p=0}^n \Omega^p(M)$ or as a map from $\Omega^p(M)$.

Lemma 2.1. *The map d^* is well defined as satisfies $d^* = (-1)^{n(p+1)+1} * d*$.*

Proof. Uniqueness: Suppose there exists two maps d_1^* and d_2^* such that for all $\alpha \in \Omega^{p-1}(M)$ and all $\beta \in \Omega^p(M)$, we have

$$(d\alpha, \beta) = (\alpha, d_1^*\beta) = (\alpha, d_2^*\beta) \Rightarrow$$
$$0 = (\alpha, d_1^*\beta - d_2^*\beta) = \int_M \langle \alpha, d_1^*\beta - d_2^*\beta \rangle * (1),$$

and one finishes the proof of uniqueness by taking $\alpha = d_1^*\beta - d_2^*\beta$. Next, we wish to show that the map $(-1)^{n(p+1)+1} * d*$ satisfies the formal adjoint property. By appealing to Stokes theorem, we have that

$$0 = \int_M d(\alpha \wedge *\beta).$$

By Lemma 1.1, and by Lemma 7.2.1 (i) of the Spring DG course, we get that

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta \\ &= d\alpha \wedge *\beta + (-1)^{p-1}(-1)^{(p-1)(n-p+1)}\alpha \wedge * * d * \beta \\ &= d\alpha \wedge *\beta - (-1)^{n(p+1)+1}\alpha \wedge * * d * \beta \end{aligned}$$

Applying $*$ to both sides of the Lemma 1.2 and appealing to Lemma 1.1 we get that the above equals

$$*(\langle d\alpha, \beta\rangle - (-1)^{n(p+1)+1}\langle \alpha, *d * \beta\rangle)$$

and thus

$$\begin{aligned} 0 = \int_M (\langle d\alpha, \beta\rangle - (-1)^{n(p+1)+1}\langle \alpha, *d * \beta\rangle * (1) \iff \\ \int_M \langle \alpha, d^*\beta\rangle * (1) = \int_M \langle \alpha, (-1)^{n(p+1)+1} * d * \beta\rangle * (1), \end{aligned}$$

which finishes the proof. □

Remark 2.1. Note that by the formal adjoint property, we have that $(d^*)^2 = 0$.

We are now ready to define the Laplace-Beltrami operator:

Definition 2.2. The Laplace-Beltrami operator on $\Omega^p(M)$ is given by

$$\Delta := dd^* + d^*d : \Omega^p(M) \to \Omega^p(M). \tag{15}$$

If $\Delta\omega = 0$ for $\omega \in \Omega^p(M)$, then ω is said to be a harmonic p-form.

Remark 2.2. To be more precise, we will in accordance with the above sometimes write

$$\Delta_p = d_{p-1}d_{p-1}^* + d_p^*d_p.$$

Corollary 2.1. *We have that Δ is formally self-adjoint, i.e $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$ for every $\alpha, \beta \in \Omega^p(M)$.*

Proof. Immediate by definition. □

Proposition 2.1. *For every $\alpha \in \Omega^p(M)$, we have that*

$$(\Delta\alpha, \alpha) = (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha). \tag{16}$$

Furthermore, Δ is non-negative and $\Delta\alpha = 0$ if and only if $d^*\alpha = 0$ and $d\alpha = 0$.

Proof. (16) follows by definition, and since the right hand side is non-negative, we get that Δ is non-negative (recall that for linear operators A and B, we say that $A \leq B$ if $(Ax, x) \leq (Bx, x)$ for all x). Next, if $\Delta\alpha = 0$, then trivially we must have that $d^*\alpha = 0 = d\alpha$. Conversely, if $d^*\alpha = 0 = d\alpha$, we get that $(\Delta\alpha, \alpha) = 0$ and by using that $\mathrm{Im}(\Delta)^\perp = \mathrm{Ker}(\Delta)$, one gets that $\Delta\alpha = 0$. □

Remark 2.3. One can view the above statement as reducing the second order differential equation $\Delta\alpha = 0$, to two first order differential equations $d\alpha = 0 = d^*\alpha$.

Corollary 2.2. *On a compact Riemannian manifold, every harmonic function is constant.*

Proposition 2.2. $*\Delta = \Delta *$

Proof. Exercise. □

2.1 Spectrum of The Laplace Operator on Forms

We continue to let M be compact and we make the reader aware of the fact that the following need to be altered slightly more to fit with the non-compact case. We note that the following notation is extremely unfortunate, because we have already used p for the degree of the form and is thus resorted to define L^k spaces for p-forms, while the more standard way would be to define L^p spaces for k-forms.

Definition 2.3. For $1 \leq k < \infty$, let $L^k_p(M)$ be the set of equivalence classes consisting of all sections of $\Lambda^p(M)$, (i.e maps $\omega : M \to \Lambda^p(M)$ with $\omega(x) \in \Lambda^p(T^*_x M)$) such that $\int_M |\omega|^k * (1) < \infty$, where we identify sections on sets of measure 0. In the case of $k = 2$, we endow it with the L^2 inner product of forms defined above to turn it into a Hilbert space. For all other k we only have a norm, namely

$$\|\omega\|_{L^k_p(M)} = \left(\int_M |\omega|^k * (1)\right)^{\frac{1}{k}}.$$

(We remind the reader that $|\omega| := \sqrt{\langle \omega, \omega \rangle}$.)

Remark 2.4. Note that since M is compact and thus a finite measure space, we have the inclusion that $L^k_p(M) \subset L^\ell_p(M)$ for all $k \geq \ell$.

We shall need the notion of a weak derivative of p-forms.

Definition 2.4. For $\omega \in L^1_p(M)$, and $0 \leq p < n$ we say that ω has a weak exterior derivative if there exists $\eta \in L^1_{p+1}(M)$ such that for all $\varphi \in \Omega^{p+1}(M)$

$$(\eta, \varphi) = (\omega, d^*\varphi)$$

and we write $d\omega = \eta$. Similarly, for $0 < p \leq n$ we say that ω has a weak d^*-derivative if there exists some $\psi \in \Omega^{p-1}(M)$ such that for all $\varphi \in \Omega^{p-1}(M)$, we have that

$$(\psi, \varphi) = (\omega, d\varphi),$$

and we write $d^*\omega = \psi$. We make the definition that for sections ω of $\Lambda^n(M)$, we have $d\omega = 0$. Similarly, for sections ω of $\Lambda^0(M)$, $d^*\omega = 0$.

Lemma 2.2. *The above definitions corresponds to the classical definitions of d and d^* if the corresponding section is smooth. Furthermore, $d^2 = 0 = (d^*)^2$ also in this setting.*

Proof. Volountary exercise. □

With this, one can define the corresponding sobolev spaces of p-forms.

Definition 2.5. For $1 \leq k < \infty$, let $W_p^{1,k}$ be the set of all $\omega \in L_p^k(M)$ such that $d\omega$, and $d^*\omega$ exists weakly and are elements of their corresponding L^k-spaces. In the case of $k = 2$, we shall denote this space by H_p and define an inner product on this space via

$$(\omega, \eta)_{H_p} := (\omega, \eta) + (d\omega, d\eta) + (d^*\omega, d^*\eta),$$

making this into a Hilbert space. For any other k, we just have a norm

$$\|\omega\|_{W_p^{1,k}(M)} := (\|\omega\|_{L_p^k(M)}^k + \|d\omega\|_{L_{p+1}^k(M)}^k + \|d^*\omega\|_{L_{p-1}^k(M)}^k)^{\frac{1}{k}}.$$

Definition 2.6. We say that $\omega \in H_p$ is a weak eigensection of Δ with eigenvalue λ if for every $\varphi \in H_p$, we have that

$$(d\omega, d\varphi) + (d^*\omega, d^*\varphi) = \lambda(\omega, \varphi).$$

Remark 2.5. For smooth sections $\omega \in \Omega^p(M)$ we could equivalently have defined eigenforms in the classical sense: $\Delta\omega = \lambda\omega$. This is eay to check.

Lemma 2.3. *Let $\pi : \Lambda^p(M) \to M$ be the footprint map, $\psi : U \to V$ be a chart, and $\varphi : \Lambda^p(M)|_{\pi^{-1}(U)} \to V \times \mathbb{R}^{\binom{n}{p}}$ be a local trivialization (i.e just a chart of the manifold $\Lambda^p(M)$. Then*

$$\|\cdot\|_{W_p^{1,k}(\psi(U))} \text{ and } \|\varphi(\cdot)\|_{W_{Eucl}^{1,k}(\psi(U))}$$

are equivalent norms on the vector space $W_p^{1,k}(\psi(U))$.

Proof. We shall skip the proof the lemma, but the idea is to use normal coordinates to around each point of U, a neighbourhood so small that $|\delta_{ij} - g_{ij}(x)| < \varepsilon$ for all $x \in U$. The special case $k = 2$ can be found as Lemma 3.4.1 in [1]. □

Remark 2.6. If one extends the above lemma to $W_p^{\alpha,k}$, for $\alpha \geq 1$, then we can transfer Sobolev space results from $\mathbb{R}^m$ locally to the manifold. In particular, we can transfer results about uniformly elliptic PDE:s on euclidean space. However, this extension to arbitrary α is not straightforward and requires the notion of the covariant derivative on tensor fields, which is something we have not yet discussed. Hence, we take it for granted and is thus provided with the following useful corollary:

Corollary 2.3. *All eigensections of* Δ *are smooth.*

In the next talk, you will probably see that we have the following orthogonal decomposition

$$L_p^2(M) = B_p \oplus B_p^* \oplus \mathcal{H}_p,$$

where B_p is the L_p^2 closure of $\{d\alpha \mid \alpha \in \Omega^{p-1}(M)\}$, B_p^* is the L_p^2 closure of $\{d^*\beta \mid \beta \in \Omega^{p+1}(M)\}$ and $\mathcal{H}_p$ is the space of smooth, harmonic p-forms.

Now, we shall discuss the spectrum of the Laplacian on p-forms. By Corollary 2.3, all eigenforms are smooth. We shall write $\sigma(\Delta)$ for the set of eigenvalues of Δ.

Lemma 2.4. *All eigenvalues of* Δ *are non-negative and eigenforms corresponding to different eigenvalues are orthogonal.*

Proof. Exercise. □

We will consider the case of positive eigenvalues. By the above decomposition of $L_p^2(M)$, we have that any eigenform $v \in \Omega^p(M)$ corresponding to a positive eigenvalue λ is contained in $B_p \oplus B_p^*$ and so we may write $v = v_1 + v_2$, where $v_1 \in B_p$ and $v_2 \in B_p^*$. The reason we get no contribution from $\mathcal{H}_p$ is that it only consists of harmonic p-forms. Now, it is not necessarily true that if the sum of two forms is smooth, that they must be smooth individually (coming up with a counterexample on $\mathbb{R}$ is easy). However, we at least have that

Lemma 2.5. *With the notation of the previous paragraph, we have that*

$$dv_1 = 0 = d^*v_2$$

Proof. For (i), note that for any $\varphi \in \Omega^{p+1}(M)$ we have that

$$(0, \varphi) = 0 = (v_1, d^*\varphi),$$

since $v_1 \in B_p \perp B_p^*$ and $d^*\varphi \in B_p^*$. Thus $dv_1 = 0$. Similarly, one shows that $d^*v_2 = 0$. □

Thus, with the above lemma, we get that

$$\lambda(v_1 + v_2) = \Delta v = (dd^* + d^*d)(v_1 + v_2) = dd^*v_1 + d^*dv_2.$$

Furthermore, we see that

$$(dd^*v_1, d^*dv_2) = (d^2d^*v_1, dv_2) = (0, dv_2) = 0,$$

and so $dd^*v_1 \perp d^*dv_2$. We make the reader aware that the above calculations are not a-priori defined, because it is not certain that d^*v_1 even exists for example. The way around this is to use the theory of distributions and thus be able to obtain the above in that sense. However, including that would make this text unnecessarily lengthy so we chose to omit it. Taking that for granted, we may conclude that

$$D_p v_1 := d_{p-1}d_{p-1}^* v_1 = \lambda v_1 \text{ and } D_p' v_2 := d_p^* d_p v_2 = \lambda v_2. \tag{17}$$

Therefore we have shown that

$$\sigma(\Delta_p) \setminus \{0\} = \sigma(D_p) \setminus \{0\} \cup \sigma(D_p') \setminus \{0\}. \tag{18}$$

To further simplify the above, we shall use the following.

Lemma 2.6. *For any two linear operators $A : X \to Y$ and $B : Y \to X$, where X and Y are normed vector spaces, we have that their (point) spectrum σ satisfies*

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$$

Proof. Take $\lambda \in \sigma(AB) \setminus \{0\}$. Then there exists $\phi \neq 0$ such that $AB\phi = \lambda\phi$. Thus, we get that

$$(BA)B\phi = B\lambda\phi = \lambda B\phi$$

Hence, $\lambda \in \sigma(BA) \setminus \{0\}$ and we may finish the proof due to symmetry reasons. □

This lemma allows us to summarize our findings in the following theorem

Theorem 2.1.

$$\sigma(\Delta_p) \setminus \{0\} = \sigma(D_{p-1}') \setminus \{0\} \cup \sigma(D_p') \setminus \{0\}$$

Proof. By using (18), together with $D_p = d_{p-1}d_{p-1}^*$, we see by Lemma 2.6 that

$$\begin{aligned}
\sigma(\Delta_p) \setminus \{0\} &= \sigma(D_p) \setminus \{0\} \cup \sigma(D_p') \setminus \{0\} \\
&= \sigma(d_{p-1}d_{p-1}^*) \setminus \{0\} \cup \sigma(D_p') \setminus \{0\} \\
&= \sigma(d_{p-1}^*d_{p-1}) \setminus \{0\} \cup \sigma(D_p') \setminus \{0\} \\
&= \sigma(D_{p-1}') \setminus \{0\} \cup \sigma(D_p') \setminus \{0\}
\end{aligned}$$

□

2.2 Consistency with other definitions

We still assume M to be compact but in these sections, only minor changes would have to be made in order to fit the non-compact case.

Now we shall perform some calculations in local coordinates to check that the Laplace operator on forms agrees with previous definitions made in the course. Recall that for $f \in C^\infty(M) = \Omega^0(M)$, we defined the Laplace-Beltrami operator, call it $\tilde{\Delta}$ (on functions) as

$$\tilde{\Delta} f = -\frac{1}{\sqrt{g}} \partial_j (\sqrt{g} g^{ij} \partial_i(f))$$

Since $d^* f = 0$, we get that the Laplace-Beltrami operator on forms, Δ, satisfies $\Delta f = d^* df$ in this case. Hence, for any test function $\varphi \in C^\infty(M)$ we may compute that

$$\begin{aligned}\int d^* df \cdot \varphi \sqrt{g} dx^1 \wedge \ldots \wedge dx^n &= (d^* df, \varphi) = (df, d\varphi) \\ &= \int \langle df, d\varphi \rangle * (1) = \int g^{ij} \partial_i(f) \partial_j(\varphi) \sqrt{g} dx^1 \wedge \ldots dx^n \\ &= -\int \partial_j(\sqrt{g} g^{ij} \partial_i(f)) \varphi dx^1 \wedge \ldots \wedge dx^n \\ &= -\int \frac{1}{\sqrt{g}} \partial_j(\sqrt{g} g^{ij} \partial_i(f)) \varphi \sqrt{g} dx^1 \wedge \ldots \wedge dx^n,\end{aligned}$$

and thus $\Delta = \tilde{\Delta}$. We saw by Gauss Theorem (Lecture "The Laplacian of Riemannian manifolds") that

$$\int_M d^* df * (1) = \int_M \Delta f * (1) = 0.$$

Which one can equivalently express as, for any *exact* 1-form ω (i.e there exists $f \in \Omega^0(M)$ such that $\omega = df$), we have that

$$\int_M d^* \omega * (1) = 0.$$

However, with what has been developed in this text, we can also conclude that for *any* 1-form ω, the above still holds. Indeed, if we let 1 denote the constant function $x \mapsto 1$, we get that

$$\int_M d^* \omega * (1) = (d^* \omega, 1) = (\omega, d1) = (\omega, 0) = 0.$$

2.3 Local calculations

Lemma 2.7. *For any $\eta \in \Omega^p(M)$, $*\eta \in \Omega^{n-p}(M)$ is the unique element satisfying*

$$\omega \wedge *\eta = \langle \omega, \eta \rangle * (1), \tag{19}$$

for all $\omega \in \Omega^p(M)$.

Proof. First, fix $\eta, \omega \in \Omega^p(M)$. By Lemma 1.2, we have that

$$*(\omega \wedge *\eta) = \langle \omega, \eta \rangle.$$

Applying $*$ to both sides and using Lemma 1.1, we get that

$$**(\omega \wedge *\eta) = *\langle \omega, \eta \rangle \Longleftrightarrow (\omega \wedge *\eta) = \langle \omega, \eta \rangle * (1),$$

And thus, we have shown that $*\eta$ satisfies (19). For uniqueness, keep η fixed and suppose there exists another such element $\tilde{\eta} \in \Omega^{n-p}(M)$ that satisfies (19) for every $\omega \in \Omega^p(M)$. Then, we get that

$$0 = \omega \wedge *\eta - \omega \wedge \tilde{\eta} = \omega \wedge *\eta - \tilde{\eta} = *\eta - \tilde{\eta} \wedge \omega.$$

In particular, with $\omega = *(*\eta - \tilde{\eta})$, we may use the previous to obtain

$$0 = *\eta - \tilde{\eta} \wedge *(*\eta - \tilde{\eta}) = \langle *\eta - \tilde{\eta}, *\eta - \tilde{\eta} \rangle$$

and thus $*\eta - \tilde{\eta} = 0$, which completes the proof. □

Remark 2.7. With this above lemma, one can compute local expressions for $*\eta$, $d^*\eta$ and ultimately $\Delta\eta$, and check that this coincides with the definition given for $d^*\eta$ for 1-forms given in the talk "The Laplacian of Riemannian manifolds". To get some inspirations for other local computations, see pages 140-143 in [1].

3 Exercises

1. Prove Lemma 1.1
2. Prove Proposition 2.2
3. Let M be a compact Riemannian manifold. Show the following statements for the Laplace operator $\Delta : \Omega^p(M) \to \Omega^p(M)$.
   (a) All eigenvalues are nonnegative.
   (b) All eigenspaces are finite dimensional.
   (c) Eigenvectors corresponding to different eigenvalues are orthogonal.

References

[1] Jürgen Jost, "Riemannian Geometry and Geometric Analysis", Springer 1995.