Cohomology classes and harmonic forms

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1 Introduction

This talk explores the representation of cohomology classes through harmonic forms. We begin by revisiting some essential definitions.

Definition 1.1. A differential form $\alpha \in \Omega^p(M)$ is called *closed* if $d\alpha = 0$, and it is called *exact* if there exists $\eta \in \Omega^{p-1}(M)$ such that $d\eta = \alpha$.

Given the operator $d : \Omega^p(M) \to \Omega^{p+1}(M)$ and its property $d \circ d = 0$ (see Lemma 7.2.11, the spring DG course), it follows that every exact forms is indeed closed.

Definition 1.2. Two closed forms $\alpha, \beta \in \Omega^p(M)$ are called *cohomologous* if $\alpha - \beta$ is exact, i.e. there exists a form $\eta \in \Omega^{p-1}(M)$ such that $\alpha - \beta = d\eta$.

This property establishes an equivalence relation on the space $\{\alpha \in \Omega^p(M) \mid d\alpha = 0\}$.

Definition 1.3. The set of equivalence classes - closed forms in $\Omega^p(M)$ modulo the exact forms - forms a vector space over $\mathbb{R}$, that is known as the p-th *de Rham cohomology group* $H^p_{dR}(M, \mathbb{R})$.

The primary goal of this talk is to prove the following theorem.

Theorem 1.4 (**Hodge**). *Let M be a compact Riemannian manifold. Then every cohomology class in $H^p_{dR}(M, \mathbb{R})$ ($0 \le p \le d = dimM$) contains precisely one harmonic form.*

The general strategy, illustrated by the Hodge theorem, is fundamental in geometric analysis. The main idea is to choose a particular representative from a class of geometric objects (here a cohomology class). The selection is achieved either by imposing a suitable differential equation or, alternatively, by minimizing a specific functional within the given class. In this case, the imposed differential equation is $d^*\eta = 0$, which, alongside the standard cohomology class equation $d\eta = 0$, results in the harmonic equation $\Delta\eta = 0$. We will illustrate the Hodge theorem using a variational method. The central technical tool will be Rellich's embedding theorem, which will be restated below in the particular form required for our purposes in Lemma 1.9.

Proof. Uniqueness. Assume we have two cohomologous harmonic differential forms $\omega_1, \omega_2 \in \Omega^p(M)$. For the case where $p = 0$, it follows that $\omega_1 = \omega_2$, since every class of $H^0_{dR}(M)$ contains just one element.

If $p > 0$ there exists a form $\eta \in \Omega^{p-1}(M)$ such that $\omega_1 - \omega_2 = d\eta$. Consequently,

$$(\omega_1 - \omega_2, \omega_1 - \omega_2) = (\omega_1 - \omega_2, d\eta) = (d^*(\omega_1 - \omega_2), \eta) = 0,$$

where, in the last step, we use the fact that ω_1 and ω_2 are harmonic, implying $d^*\omega_1 = 0$ and $d^*\omega_2 = 0$ (see Proposition 2.1, Talk 7).

Since the scalar product is positive definite, we conclude that $\omega_1 = \omega_2$ which implies uniqueness. □

To establish the existence, a more challenging task, we will use Dirichlet's principle. Let ω_0 be a closed differential form, representing a given cohomology class in $H^p(M)$. Then any form cohomologous to ω_0 can be written as

$$\omega = \omega_0 + d\alpha \quad \text{with } \alpha \in \Omega^{p-1}(M).$$

Now we minimise the L^2-norm $D(\omega) = (\omega, \omega)$ in the class of all forms cohomologous to ω_0. Assume that the infimum is achieved by a smooth form η, and let $\eta + td\beta$ with $\beta \in \Omega^{p-1}(M)$ be a variation of that form. Then η has to satisfy the following Euler–Lagrange equations for D

$$\begin{aligned} 0 &= \frac{d}{dt}(\eta + td\beta, \eta + td\beta)_{|t=0} \\ &= 2(\eta, d\beta) \\ &= 2(d^*\eta, \beta) \end{aligned} \tag{1}$$

which implies $d^*\eta = 0$, and hence the harmonicity of η.

Since our objective is to minimize the L^2-norm, we need a space that is complete with respect to L^2-convergence. Thus, we have to work with the space of L^2 forms, instead of the space of the smooth forms. For technical reasons, it is necessary to define the Sobolev space in the current context.

Definition 1.5. Let E be a vector bundle over M and $s : M \to E$ a section of E with compact support. A section s is contained in the *Sobolev space* $H^{k,r}(E)$, if for any bundle atlas with the property that on compact sets all coordinate changes and all their derivatives are bounded and for any bundle chart from such an atlas

$$\varphi : E_{|U} \to U \times \mathbb{R}^n$$

we have that $\varphi \circ s_{|U}$ is contained in $H^{k,r}(E)$

Remark 1.6. *It is possible to obtain such an atlas by making coordinate neighborhood smaller if necessary*

Consider a new scalar product on $\Omega^p(M)$

$$((\omega,\omega)) := (d\omega, d\omega) + (d^*\omega, d^*\omega) + (\omega,\omega)$$

and the norm

$$\|\omega\|_{H^{1,2}(M)} := ((\omega,\omega))^{\frac{1}{2}}.$$

We consider a completion of the space $\Omega^p(M)$ of smooth p-forms with respect to the $\|\cdot\|_{H^{1,2}(M)}$-norm. The resulting Hilbert space will be denoted as $H_p^{1,2}(M)$, or simply $H^{1,2}(M)$, when the index p is clear from the context.

Recall that for a given open set $V \subset \mathbb{R}^d$ and a smooth map $f : V \to \mathbb{R}^n$, the Euclidean Sobolev norm is given by

$$\|f\|_{H^{1,2}_{\text{eucl.}}(V)} := \left(\int_V f \cdot f + \int_V \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_i} \right)^{\frac{1}{2}}$$

For every $x_0 \in M$ there exists an open neighborhood U and a diffeomorphism

$$\varphi : \Lambda^p(M)_{|U} \to V \times \mathbb{R}^n$$

where V is open in $\mathbb{R}^d$ and n is the dimension of the fibers of $\Lambda^p(M)$ and it equals $\binom{d}{p}$. We also define the projection $\pi : V \times \mathbb{R}^n \to V$ which maps the fiber over $x \in U$ to a fiber $\{\pi(\varphi(x))\} \times \mathbb{R}^n$.

Lemma 1.7. *On any subset U' of U with $\overline{U'} \subset U$ and $V' = \pi(\varphi(U'))$, the norms*

$$\|\omega\|_{H^{1,2}(U')} \quad \textit{and} \quad \|\varphi(\omega)\|_{H^{1,2}_{eucl.}(V')}$$

are equivalent.

Proof. If we restrict ourselves to relatively compact subsets U' of U, we know that all coordinate changes lead to equivalent norms. Then, it is enough to find for every x in $\overline{U'}$ a neighborhood U'' on which the norms are equivalent, since the claim then follows by a covering argument.

Recall that the local coordinates defined by the chart $(\exp_p^{-1}, U)$ are called *normal coordinates* with center p. We also will need the following result.

Theorem 1.8. *In normal coordinates, we have for the Riemannian metric $g_{ij}(0) = \delta_{ij}$, $\Gamma^i_{jk}(0) = 0$ for all i, j, k.*

We can assume that $\pi \circ \varphi$ is the map onto normal coordinates with center x_0, thus by continuity there exists $\varepsilon > 0$ such that in the neighborhood U'' of x_0 we have

$$|g_{ij}(x) - \delta_{ij}| \le \varepsilon, \quad \text{for } i,j = 1,..,d, \tag{2}$$

$$|\Gamma^i_{jk}(x)| \le \varepsilon \quad \text{for } i,j,k = 1,..,d, \tag{3}$$

To continue we will need first to obtain some quantities.

Let $\alpha, \beta \in \Omega^p(M)$ and write them as

$$\alpha = \alpha_{i_1..i_p} dx^{i_1} \wedge ... \wedge dx^{i_p},$$
$$\beta = \beta_{j_1..j_p} dx^{j_1} \wedge ... \wedge dx^{j_p}.$$

then

$$< \alpha, \beta >= \alpha_{i_1..i_p} \beta_{j_1..j_p} g^{i_1 j_1} g^{i_2 j_2} ... g^{i_p j_p}. \tag{4}$$

Recall that

$$d\alpha = \sum_{k=1}^{d-p} \frac{\partial \alpha_{i_1..i_p}}{\partial x^{j_k}} dx^{j_k} \wedge dx^{i_1} \wedge ... \wedge dx^{i_p}.$$

Then

$$< d\alpha, d\beta >= \frac{\partial \alpha_{i_1..i_p}}{\partial x^k} \frac{\partial \beta_{j_1..j_p}}{\partial x^l} g^{kl} g^{i_1 j_1} g^{i_2 j_2} ... g^{i_p j_p} \tag{5}$$

Recall (from the talk 5) that if $\omega \in \Omega^1(M)$ we have that

$$d^* \omega = -g^{kl} \left(\frac{\partial \omega_k}{\partial x_l} - \Gamma^j_{kl} \omega_k \right).$$

Similarly,

$$(d^* \alpha)_{i_1..i_{p-1}} = -g^{kl} \left(\frac{\partial \alpha_{k i_1..i_{p-1}}}{\partial x^l} - \Gamma^j_{kl} \alpha_{j i_1..i_{p-1}} \right).$$

Then

$$\begin{aligned} < d^*\alpha, d^*\beta > =& < g^{kl} \left(\frac{\partial \alpha_{k i_1..i_{p-1}}}{\partial x^l} - \Gamma^j_{kl} \alpha_{j i_1..i_{p-1}} \right) dx^{i_1} \wedge ... \wedge dx^{i_p}, \\ & g^{mn} \left(\frac{\partial \beta_{m j_1..j_{p-1}}}{\partial x^n} - \Gamma^r_{mn} \beta_{r j_1..j_{p-1}} \right) dx^{j_1} \wedge ... \wedge dx^{j_p} > \\ =& \frac{\partial \alpha_{k i_1..i_{p-1}}}{\partial x^l} \frac{\partial \beta_{m j_1..j_{p-1}}}{\partial x^n} g^{kl} g^{mn} g^{i_1 j_1} ... g^{i_{p-1} j_{p-1}} \\ & - \frac{\partial \alpha_{k i_1..i_{p-1}}}{\partial x^l} \Gamma^r_{mn} \beta_{r j_1..j_{p-1}} g^{kl} ... g^{i_{p-1} j_{p-1}} \\ & - \frac{\partial \beta_{m j_1..j_{p-1}}}{\partial x^n} \Gamma^j_{kl} \alpha_{j i_1..i_{p-1}} g^{kl} g^{mn} g^{i_1 j_1} ... g^{i_{p-1} j_{p-1}} \\ & + \Gamma^j_{kl} \Gamma^r_{mn} \alpha_{j i_1..i_{p-1}} \beta_{r j_1..j_{p-1}} g^{kl} ... g^{i_{p-1} j_{p-1}}. \end{aligned} \tag{6}$$

Recall, that we actually considered the following L^2-product as following

$$(\alpha, \beta) = \int_M < \alpha, \beta > *(1), \quad \alpha, \beta \in \Omega^p(U)$$

Then set

$$\omega = \omega_{i_1..i_p} dx^{i_1} \wedge ... \wedge dx^{i_p}$$

and observe that

$$\|\varphi(\omega)\|_{H^{1,2}_{eucl}(V'')} = \int_{V''} \omega_{i_1..i_p} \cdot \omega_{j_1..j_p} dx^1...dx^d + \int_{V''} \frac{\partial \omega_{i_1..i_p}}{\partial x_i} \cdot \frac{\partial \omega_{j_1..j_p}}{\partial x_i} dx^1...dx^d$$
$$= \int_{V''} \omega_{i_1..i_p} \cdot \omega_{j_1..j_p} \delta^{i_1 j_1}...\delta^{i_p j_p} dx^1...dx^d$$
$$+ \int_{V''} \frac{\partial \omega_{i_1..i_p}}{\partial x_i} \cdot \frac{\partial \omega_{i_1..i_p}}{\partial x_j} \delta^{ij} \delta^{i_1 j_1}...\delta^{i_p j_p} dx^1...dx^d$$

where $V'' = \pi(\varphi(U''))$. Now using the estimates (2), (3) and equalities (4), (5), for some $c > 0$ we obtain

$$c\,\|\varphi(\omega)\|_{H^{1,2}_{eucl}(V'')} \leq \int_{V''} < \omega, \omega > \sqrt{g} dx^1...dx^d + \int_{V''} < d\omega, d\omega > \sqrt{g} dx^1...dx^d$$
$$= (\omega, \omega) + (d\omega, d\omega) \leq (\omega, \omega) + (d\omega, d\omega) + (d^*\omega, d^*\omega) = \|\omega\|_{H^{1,2}(U'')}$$

To get the upper bound we proceed in the similar way, using the aforementioned estimates and equalities plus equality (6). In particular, to bound $(d^*\omega, d^*\omega)$ with $(d\omega, d\omega)$ and (ω, ω) we just use inequalities of the following type

$$\frac{\partial \alpha_{k i_1 \ldots i_{p-1}}}{\partial x^l} \beta_{r j_1 \ldots j_{p-1}} \leq \frac{1}{2} \left(\left| \frac{\partial \alpha_{k i_1 \ldots i_{p-1}}}{\partial x^l} \right|^2 + |\beta_{r j_1 \ldots j_{p-1}}|^2 \right)$$

Then the claim holds for sufficiently small $\varepsilon > 0$, that is for a sufficiently small neighborhood of x_0. The claim follows by a covering argument □

With the help of Lemma 1.7 the results for Sobolev spaces in the Euclidean settings can be extended to a Riemannian manifold. In particular, the following result holds.

Lemma 1.9 (Rellich's theorem)**.** *Let $(\omega_n)_{n\in\mathbb{N}} \subset H^{1,2}_p(M)$ be bounded, i.e.*

$$\|\omega_n\|_{H^{1,2}(M)} \leq K.$$

Then a subsequence of (ω_n) converges w.r.t. the L^2-norm

$$\|\omega\|_{L^2(M)} := (\omega, \omega)^{\frac{1}{2}}$$

to some $\omega \in H^{1,2}_p(M)$.

Corollary 1.10. *There exists a constant c, depending only on the Riemanian metric of M, with the property that for all closed forms β that are orthogonal to the kernel of d^*,*

$$(\beta, \beta) \leq c(d^*\beta, d^*\beta) \tag{7}$$

Proof. We prove by contradiction. Assume there exists a sequence of closed forms β_n orthogonal to the kernel of d^* such that

$$(\beta_n, \beta_n) \geq n(d^*\beta_n, d^*\beta_n). \tag{8}$$

Define

$$\lambda_n := (\beta_n, \beta_n)^{-\frac{1}{2}}.$$

Multiplying both sides of (8) by λ_n, we obtain

$$(d^*(\lambda_n\beta_n), d^*(\lambda_n\beta_n)) \leq \frac{1}{n}(\lambda_n\beta_n, \lambda_n\beta_n) = \frac{1}{n}. \tag{9}$$

Since $d\beta_n = 0$, we have

$$\begin{aligned} \|\lambda_n\beta_n\|_{H^{1,2}} &= (\lambda_n\beta_n, \lambda_n\beta_n) + \lambda_n^2(d\beta_n, d\beta_n) + (d^*(\lambda_n\beta_n), d^*(\lambda_n\beta_n)) \\ &= 1 + (d^*(\lambda_n\beta_n), d^*(\lambda_n\beta_n)) \leq 1 + \frac{1}{n}. \end{aligned}$$

By Lemma 1.9, up to selecting a susequence, $\lambda_n\beta_n$ converges in L^2-norm to some form ψ. By (9) $d^*(\lambda_n\beta_n)$ converges to 0 in L^2, and so for any φ

$$\begin{aligned} (d^*\psi, \varphi) = (\psi, d\varphi) &= \lim_{n\to\infty}(\lambda_n\beta_n, d\varphi) \\ &= \lim_{n\to\infty}(d^*(\lambda_n\beta_n), \varphi) = 0 \end{aligned}$$

and hence $d^*\psi = 0$.

And again, since by assumption β_n are closed, we have

$$\begin{aligned} (d\psi, \varphi) = (\psi, d^*\varphi) &= \lim_{n\to\infty}(\lambda_n\beta_n, d^*\varphi) \\ &= \lim_{n\to\infty}(d(\lambda_n\beta_n), \varphi) = 0 \end{aligned} \tag{10}$$

and $d\psi = 0$.

Since $d^*\psi = 0$ (so it belongs to the kernel of d^*) and β_n is orthogonal to the kernel of d^* by assumption

$$(\psi, \lambda_n\beta_n) = 0.$$

Recall that $(\lambda_n\beta_n, \lambda_n\beta_n) = 1$, then

$$\lim_{n\to\infty}(\psi, \lambda_n\beta_n) = 1,$$

which contradicts to (10). □

Proof. Existence part of Theorem 1.4. We begin by roughly outlining the idea of the proof. We minimize the functional $D(\omega) = (\omega, \omega)$ for ω in the cohomology class we consider. Then we pick a ω that minimizes $D(\omega)$. Then, using (1) we get that, $(d^*\omega, \beta) = 0$, which together with the assumption $d\omega = 0$ implies that ω is harmonic.

However, there are some important details being missed when doing this. Firstly, we don't know whether the ω that minimizes $D(\omega)$ is in the cohomology class. Indeed, we will construct ω as a weak limit point, and (1) will then give us that $(\omega, d\beta) = 0$ for all $\beta \in \Omega^{p-1}(M)$, which means that ω is weakly harmonic. And due to the Laplacian being a uniformly elliptic operator we get by regularity theory that ω is smooth and harmonic in the usual sense. But we also need to assure that the ω we construct as a weak limit does not leave the considered cohomology class. Also in order to use (1) we need to ensure that $\omega + dd\beta$ does not leave the set we are minimizing over, so we need to minimize D over the set of weak limit points of the cohomology class we consider.

We now carry out the rigorous argument. Let C_{ω_0} be the set of all weak limit points of forms in the cohomology class of $\omega_0 \in \Omega^p(M)$, i.e.

$$C_{\omega_0} = \{\omega \in L^2 | \ \exists (\alpha_n)_{n=1}^{\infty} \subset \Omega^{p-1}(M) : \lim_{n\to\infty} (\omega_0 + d\alpha_n, \varphi) = (\omega, \varphi) \ \forall \varphi \in \Omega^p(M)\}.$$

We shall now classify C_{ω_0} as all ω being weakly in the same cohomology class as ω_0, i.e. we shall show that for all $\omega \in C_{\omega_0}$ there exists $\alpha \in L^2$ such that

$$(\alpha, d^*\varphi) = (\omega - \omega_0, \varphi) \ \forall \varphi \in \Omega^p(M). \tag{11}$$

First we note that for $\omega \in C_{\omega_0}$ we have

$$(\omega - \omega_0, \varphi) = 0 \tag{12}$$

for all φ such that $d^*\varphi = 0$, since we have $(\omega - \omega_0, \varphi) = \lim_{n\to\infty}(d\alpha_n + \omega_0 - \omega_0, \varphi) = \lim_{n\to\infty}(\alpha_n, d^*\varphi) = 0$ if $d^*\varphi = 0$. Now, let $\eta := \omega - \omega_0$, where $\omega \in C_{\omega_0}$. Define a linear functional l on $d^*(\Omega^p(M))$ by

$$l(d^*\varphi) := (\eta, \varphi).$$

Since d^* and the scalar product is linear, l is linear. Also l is well-defined since $d^*\varphi_1 = d^*\varphi_2$ implies $d^*(\varphi_1 - \varphi_2) = 0$ and so by (12) we have that $l(d^*\varphi_1) = l(d^*\varphi_2)$.

Not define $\pi : \Omega^p(M) \to \ker d_p^*$ as the projection onto the kernel of d^*, and let $\varphi \in \Omega^p(M)$ and consider $\psi := \varphi - \pi(\varphi)$. By definition of π we have $d^*\varphi = d^*\psi$. So we have

$$l(d^*\varphi) = l(d^*\psi) = (\eta, \psi). \tag{13}$$

Because ψ is orthogonal to $\ker d^*$ we have by Corollary 1.10 that

$$\|\psi\|_{L^2} \le c\, \|d^*\psi\|_{L^2} = c\, \|d^*\varphi\|_{L^2}. \tag{14}$$

Together, (13), (14) and Cauchy-Schwartz imply

$$|l(d^*\varphi)| \le c\, \|\eta\|_{L^2} \|d^*\varphi\|_{L^2}.$$

So l is bounded on $d^*(\Omega^p(M))$ and we can extend l continuously to the L^2 closure of $d^*(\Omega^p(M)$, which then becomes a Hilbert space. Denote the extension of l again by l. Since l is now a bounded linear funcitonal on a Hilbert space, we can use Riesz representation theorem, which gives us an $\alpha \in \overline{d^*(\Omega^p(M)}$ such that

$$(\alpha, d^*\varphi) = (\eta, \varphi)$$

for all $\varphi \in \Omega^p(M)$. This is exactly (11).

Now that we have (11) we are ready to wrap up the proof. Consider

$$\kappa := \inf_{\omega \in C_{\omega_0}} D(\omega) \tag{15}$$

for an arbitrary closed $\omega_0 \in \Omega^p(M)$. Since $D(\omega) = \|\omega\|_{L^2}^2$ we can choose a minimizing sequence of (15) $(\omega_n)_{n=1}^\infty$ and may assume that $\|\omega_n\|_{L^2} \leq \kappa + 1$ for all n. But a bounded sequence in a Hilbert space always has a subsequence that converges weakly. Denote the weak limit by ω, hence we may assume that (ω_n) converges weakly to ω, i.e. $(\omega_n, \varphi) = (\omega, \varphi)$ for all $\varphi \in \Omega^p(M)$. Also, since C_{ω_0} is the closure with respect to weak limit points, ω is contained in C_{ω_0}.

Now we use that D is weakly lower semicontinuous with respect to weak convergence

$$\kappa \leq D(\omega) \leq \liminf_{n\to\infty} D(\omega_n) = 0,$$

which gives us that $D(\omega) = \kappa$. Using this fact, we are ready to use (1) with ω in place of η, since $\omega + d\beta \in C_{\omega_0}$ (this is easily seen from the classification (11)) and $D(\omega) = \kappa$. Of course, we can not carry out the last equality in (1), since we do not yet know that ω is equivalent to a smooth form. But we have $(\omega, d\beta) = 0$ for all $\beta \in \Omega^{p-1}(M)$. Also, we have that $(\omega, d^*\varphi) = 0$ for all $\varphi \in \Omega^{p+1}(M)$, since by (11) we have $(\omega, d^*\varphi) = (\omega - \omega_0, d^*\varphi) + (\omega_0, d^*\varphi) = (\alpha, d^*d^*\varphi) + 0 = 0$. Together, $(\omega, d\beta) = (\omega, d^*\varphi) = 0$ means that ω is weakly harmonic. Hence ω is a weak solution to Laplace's equation. And since the Laplace operator is elliptic, we get by regularity theory that ω is smooth and harmonic in the usual sense. For details, see [1, A.2].

Now we are almost done. We have shown that there exists a harmonic $\omega \in \Omega^p(M)$ which is weakly in the same cohomology class as ω_0, i.e. (11) holds. But we still need to show that ω is in the same cohomology class as ω_0, i.e. that α in (11) is smooth.[1] When we have proved this, we are done. We shall use the Sobolev embedding theorem which reduces the problem of showing that α is smooth, to showing that it is contained in $H^{k,2}$ for all $k \in \mathbb{N}$. For details see [1, A.1]. But this is not so difficult to see. We have $(\alpha, d^*\varphi) = (\omega - \omega_0, \varphi)$. This means that $d\alpha$ exists, weakly, and is smooth (it is equal to $\omega - \omega_0$). Hence all other derivatives (starting with d) will exist and be bounded. Also $(\alpha, d\beta) = 0$ for all $\beta \in \Omega^{p-1}(M)$, since $\alpha \in \overline{d^*(\Omega^p(M)}$, and derivation conserves weak convergence. So weakly $d^*\alpha = 0$. Hence all weak derivatives of α exists and are continuous, and since we are on a compact manifold, they are contained in $H^{k,2}$ for arbitrary k and we are done. □

Corollary 1.11. *Let B_p be the L^2-closure of $\{d\alpha : \alpha \in \Omega^{p-1}(M)\}$, and B_p^* be the L^2-closure of $\{d^*\beta : \beta \in \Omega^{p+1}(M)\}$, and $\mathcal{H}_p$ be the set of harmonic p-forms.*

[1]This is equivalent to showing that $C_{\omega_1} \cap C_{\omega_2} = \emptyset$ for ω_1, ω_2 in different cohomology classes.

Then the Hilbert space L_p^2 consisting of square inteagrable p-forms admits the orthogonal decomposition

$$L_p^2(M) = B_p \oplus B_p^* \oplus \mathcal{H}_p.$$

Moreover, we have $\mathcal{H}_p = B_p^\perp \cap B_p^{*\perp}$.

Proof. First we note that $\{d\alpha : \alpha \in \Omega^{p-1}(M)\}$ and $\{d^*\beta : \beta \in \Omega^{p+1}(M)\}$ are orthogonal to each other, since $(d\alpha, d^*\beta) = (d^2\alpha, \beta) = 0$, and by a standard continuity argument ($(\cdot, \omega)$ is continuous), we get that B_p and B_p^* are orthogonal. From this fact we have that

$$L_p^2(M) = B_p \oplus B_p^* \oplus \left(B_p^\perp \cap B_p^{*\perp}\right).$$

So all that is left is to prove that $\mathcal{H}_p = B_p^\perp \cap B_p^{*\perp}$. But $\omega \in B_p^\perp \cap B_p^{*\perp}$ implies $(\omega, d\alpha) = 0$ and $(\omega, d^*\beta) = 0$ for all $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^{p+1}(M)$. And the last step of Theorem 1.4 was proving that this implies that ω is smooth and harmonic, so we are done. ☐

Corollary 1.12. *Let M be a compact differentiable manifold. Then all cohomology groups $H^p_{dR}(M, \mathbb{R})$ $(0, \leq p \leq d := \dim m)$ are finite dimensional.*

Proof. Firstly, since M is a differentiable manifold, we may equip it with a Riemannian metric. And by Theorem 1.4 we know that each cohomology class may be represented by a form which is harmonic with respect to the chosen metric. Due to this fact, we may define a scalar product on $H^p_{dR}(M, \mathbb{R})$ by $([u], [v])_{H^p} := (u_0, v_0)_{L^2}$, where u_0 and v_0 are the unique harmonic representatives of the equivalence class. If we assume $H^p_{dR}(M, \mathbb{R})$ to be infinitely dimensional there therefore must exist an orthonormal sequence of harmonic forms $(\eta_n)_{n\in\mathbb{N}}$ w.r.t. $(\cdot, \cdot)_{L^2}$. I.e. we have $(\eta_n, \eta_m) = \delta_{nm}$. But since η_n are harmonic, they satisfy $d\eta_n = d^*\eta_n = 0$, so we have $\|\eta_n\|_{H^{1,2}(M)} = 1$. So Rellich's theorem (Lemma 1.9) gives us a subsequence of (η_n) that converges in L^2, which contradicts the orthogonality assumption. ☐

References

[1] Jost, Jürgen. *Riemannian geometry and geometric analysis.* Springer, 2017.