# Introduction to Macdonald polynomials: Lecture 1 

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## 0 Introduction

Overview of Macdonald polynomials $P_{\lambda}(x ; q, t)$ in $n$ variables $x=\left(x_{1}, \ldots, x_{n}\right)$ (of type $\left.A_{n-1}\right)$.

### 0.1 Three monographs by I.G. Macdonald

[1] Symmetric Functions and Hall Polynomials, Second Edition. Oxford University Press, 1995, $\mathrm{x}+475 \mathrm{pp}$.

- Chapter VI: Symmetric functions with two paramters $P_{\lambda}(x ; q, t)$.
[2] Symmetric Functions and Orthogonal Polynomials. University Lecture Series 12, American Mathematical Society, 1998, xvi+53 pp.
[3] Affine Hecke Algebras and Orthogonal Polynomials. Cambridge Tracts in Mathematics 157, Cambridge University Press, 2003, x+175 pp.
- Macdonald-Cherednik theory based on (double) affine Hecke algebras)


### 0.2 Macdonald polynomials

Macdonald polynomials ( $\sim 1987$ )

- in the narrow sense:
symmetric polynomials $P_{\lambda}(x ; q, t)$ in $x=\left(x_{1}, \ldots, x_{n}\right)$ of [1] (of type $A_{n-1}$ in the $\mathrm{GL}_{n}$ version)
- in the broader sense:
$W$-invariant orthogonal polynomials with parameters $q$ and $\left(t_{\alpha}\right)_{\alpha}$ associated with a root system (a pair of a root system and a lattice on which the Weyl group $W$ acts)
The Macdonald polynomials of type $B C$ (or $C^{\vee} C$ ) are also called the Koornwinder polynomials.
The main objects of this course: Macdonald polynomials in $n$ variables (of type $A, \mathrm{GL}_{n}$ version)

$$
\begin{equation*}
P_{\lambda}(x ; q, t) \in \mathbb{C}[x]^{\mathfrak{G}_{n}}, \quad x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) . \tag{0.1}
\end{equation*}
$$

- a family of $\mathbb{C}$-bases of the ring of symmetric polynomials with two parameters $(q, t)$.

The ring $\mathbb{C}[x]^{\mathfrak{S}_{n}}$ of symmetric polynomials $x=\left(x_{1}, \ldots, x_{n}\right)$ have two fundamental bases

$$
\begin{equation*}
\mathbb{C}[x]^{\mathfrak{G}_{n}}=\bigoplus_{\lambda \in \mathcal{P}_{n}} \mathbb{C} m_{\lambda}(x)=\bigoplus_{\lambda \in \mathcal{P}_{n}} \mathbb{C} s_{\lambda}(x) \tag{0.2}
\end{equation*}
$$

These bases are indexed by the set $\mathcal{P}_{n}$ of partitions $\lambda$ with $\ell(\lambda) \leq n$ :

$$
\begin{equation*}
\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right) \in \mathbb{N}^{n}, \quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n} \geq 0 \tag{0.3}
\end{equation*}
$$

The symmetric polynomials

$$
\begin{equation*}
m_{\lambda}(x)=\sum_{\mu \in \mathfrak{S}_{n} . \lambda} x^{\mu}=x^{\lambda}+\cdots, \quad s_{\lambda}(x)=\frac{\operatorname{det}\left(x_{i}^{\lambda_{j}+n-j}\right)_{i, j=1}^{n}}{\operatorname{det}\left(x_{i}^{n-j}\right)_{i, j=1}^{n}}=x^{\lambda}+\cdots \tag{0.4}
\end{equation*}
$$

are called the monomial symmetric functions (orbit sums) and the Schur functions, respectively. Both $m_{\lambda}(x)$ and $s_{\lambda}(x)$ have leading term $x^{\lambda}=x_{1}^{\lambda_{1}} \cdots x_{n}^{\lambda_{n}}$ with respect to a partial order of partitions, called the dominance order.

The Macdonald polynomials $P_{\lambda}(x)=P_{\lambda}(x ; q, t)$, which will be specified below, provide a family of bases of $\mathbb{C}[x]^{\mathfrak{G}_{n}}$ with parameters $(q, t)$; they specialize to $m_{\lambda}(x)$ when $t=1$, and to $s_{\lambda}(x)$ when $t=q$. Also, in the limit as $q \rightarrow 1$ with scaling $t=q^{\beta}$, they recover the Jack polynomials $P_{\lambda}^{(\beta)}(x)$. Other two important special cases are the Hall-Littlewood polynomials $P_{\lambda}(x ; t)=P_{\lambda}(x ; 0, t)$ and the $q$-Whittaker functions $P_{\lambda}(x ; q, 0)$.

Consider the Macdonald-Ruijsenaars $q$-difference operator

$$
\begin{equation*}
D_{x}=\sum_{i=1}^{n} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}} T_{q, x_{i}}=\prod_{j=2}^{n} \frac{t x_{1}-x_{j}}{x_{1}-x_{j}} T_{q, x_{1}}+\cdots \tag{0.5}
\end{equation*}
$$

where $T_{q, x_{i}}$ stands for the $q$-shift operator with respect to the variable $x_{i}$ :

$$
\begin{equation*}
T_{q, x_{i}} f\left(x_{1}, \ldots, x_{i}, \ldots, x_{n}\right)=f\left(x_{1}, \ldots, q x_{i}, \ldots, n\right) . \tag{0.6}
\end{equation*}
$$

Theorem (Macdonald) For each partition $\lambda \in \mathcal{P}_{n}$ with $\ell(\lambda) \leq n$, there exists a unique symmetric polynomial $P_{\lambda}(x)=P_{\lambda}(x ; q, t) \in \mathbb{Q}(q, t)[x]^{\mathfrak{G}_{n}}$ such that
(1) $\quad D_{x} P_{\lambda}(x)=d_{\lambda} P_{\lambda}(x), \quad d_{\lambda}=\sum_{i=1}^{n} t^{n-i} q^{\lambda}$,
(2) $\quad P_{\lambda}(x)=m_{\lambda}(x)+$ (lower order terms with respect the dominance order).

We remark that the Jack polynomials $P_{\lambda}^{(\beta)}(x)$ are orthogonal polynomials associated with the Heckman-Opdam system/Calogero-Sutherland system of type $A_{n-1}$. They are the polynomial joint eigenfunctions of a commuting family of differential operators, called the Sekiguchi operators. The Macdonald polynomials are also the orthogonal polynomials (polynomial joint eigenfunctions) associated with the commuting family of Macdonald-Ruijsenaars $q$-difference operators, which define a difference version (relativistic version) of the differential (non-relativistic) Calogero-Sutherland system.

### 0.3 Fundamental properties of Macdonald polynomials

(a) Specializations: Schur, Jack, Hall-Littlewood, $q$-Whittacker.
(b) $\boldsymbol{q}$-Difference equations: There exists a commuting family of higher order $q$-difference operators $D_{x}^{(1)}, \ldots, D_{x}^{(n)}$ starting from $D_{x}^{(1)}=D_{x}$, for which the Macdonald polynomials are joint eigenfunctions:

$$
\begin{align*}
& D_{x}^{(r)}=\sum_{\substack{I \subseteq\{1, \ldots, n\} \\
|I|=r}} t^{\binom{r}{2}} \prod_{i \in I ; j \notin I} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}} \prod_{i \in I} T_{q, x_{i}} \quad(r=1, \ldots, n),  \tag{0.8}\\
& D_{x}^{(r)} P_{\lambda}(x)=e_{r}\left(t^{\delta} q^{\lambda}\right) P_{\lambda}(x) \quad\left(\lambda \in \mathcal{P}_{n}, r=1, \ldots, n\right),
\end{align*}
$$

where the eigenvalues $e_{r}\left(t^{\delta} q^{\lambda}\right)$ are $r$ th symmetric functions in $t^{\delta} q^{\lambda}=\left(t^{n-1} q^{\lambda_{1}}, t^{n-2} q^{\lambda_{2}}, \ldots, q^{\lambda_{n}}\right)$.
(c) Orthogonality: The Macdonald polynomials are orthogonal polynomials on the torus $\mathbb{T}^{n}=$ $\left\{\left|x_{1}\right|=\cdots=\left|x_{n}\right|=1\right\}$ with respect to the scalar product define by the weight function

$$
\begin{equation*}
w(x ; q, t)=\prod_{1 \leq i<j \leq n} \frac{\left(x_{i} / x_{j} ; q\right)_{\infty}}{\left(t x_{i} / x_{j} ; q\right)_{\infty}} \frac{\left(x_{j} / x_{i} ; q\right)_{\infty}}{\left(t x_{j} / x_{i} ; q\right)_{\infty}}, \quad(z ; q)_{\infty}=\prod_{i=0}^{\infty}\left(1-q^{i} z\right) \quad(|q|<1) \tag{0.9}
\end{equation*}
$$

which is a $q$-version of $w^{(\beta)}(x)=\prod_{1 \leq i<j \leq n}\left|x_{i}-x_{j}\right|^{2 \beta}$ in the case of Jack polynomials. - constant term conjecture and the scalar product conjecture.
(d) Evaluation formula and self-duality: The value $P_{\lambda}\left(t^{\delta}\right)$ at the base point $x=t^{\delta}=$ $\left(t^{n-1}, t^{n-2}, \ldots, 1\right)$ can be evaluated explicitly. Normalize $P_{\lambda}(x)$ as $\widetilde{P}_{\lambda}(x)=P_{\lambda}(x) / P_{\lambda}\left(t^{\delta}\right)$, so that $\widetilde{P}_{\lambda}\left(t^{\delta}\right)=1$. Then we have the symmetry $\widetilde{P}_{\lambda}\left(t^{\delta} q^{\mu}\right)=\widetilde{P}_{\mu}\left(t^{\delta} q^{\lambda}\right)\left(\lambda, \mu \in \mathcal{P}_{n}\right)$ with respect to the position variables $x=t^{\delta} q^{\mu}$ and the spectral variables $\xi=t^{\delta} q^{\lambda}$.
(e) Branching rules and Pieri formula
(f) Generating functions: of Cauchy type and of dual Cauchy type.
(g) Integrality of coefficients: Integral form $J_{\lambda}(x) \in \mathbb{Z}[q, t][x]^{\mathfrak{S}_{n}}, J_{\lambda}(x)=c_{\lambda} P_{\lambda}(x)$.
(h) Double affine Heck algebras: Dunkl operators and nonsymmetric Macdonald polynomials.

## 1 Schur functions

### 1.1 Symmetric polynomials $e_{r}(x), h_{l}(x)$ and $p_{k}(x)$

$e_{r}(x)$ : elementary symmetric functions, $h_{l}(x)$ : complete homogeneous symmetric functions

$$
\begin{align*}
& e_{r}(x)=\sum_{1 \leq i_{1}<\cdots<i_{r} \leq n} x_{i_{1}} \cdots x_{i_{r}} \quad(r=0,1,2, \ldots), \quad e_{r}(x)=0 \quad(r>n), \\
& h_{l}(x)=\sum_{\mu_{1}+\cdots+\mu_{n}=l} x_{1}^{\mu_{1}} \cdots x_{n}^{\mu_{n}}=\sum_{1 \leq j_{1} \leq \cdots \leq j_{l} \leq n} x_{j_{1}} x_{j_{2}} \cdots x_{j_{l}} \quad(l=0,1,2, \ldots) . \tag{1.1}
\end{align*}
$$

$p_{k}(x):$ power sums

$$
\begin{equation*}
p_{k}(x)=x_{1}^{k}+x_{2}^{k}+\cdots+x_{n}^{k} \quad(k=1,2, \ldots) . \tag{1.2}
\end{equation*}
$$

- Generation functions, Newton formula, Wronski formula
- The ring of symmetric polynomials in $x=\left(x_{1}, \ldots, x_{n}\right)$ :

$$
\begin{equation*}
\mathbb{C}[x]^{\mathfrak{S}_{n}}=\left\{f \in \mathbb{C}[x] \mid \sigma(f)=f \quad\left(\sigma \in \mathfrak{S}_{n}\right)\right\}=\mathbb{C}\left[e_{1}(x), \ldots, e_{n}(x)\right] . \tag{1.3}
\end{equation*}
$$

The elementary symmetric functions $e_{1}(x), \ldots, e_{n}(x)$ are algebraically independent over $\mathbb{C}$. In this $n$ variable case, we also have

$$
\begin{equation*}
\mathbb{C}[x]^{\mathfrak{S}_{n}}=\mathbb{C}\left[h_{1}(x), \ldots, h_{n}(x)\right]=\mathbb{C}\left[p_{1}(x), \ldots, p_{n}(x)\right] . \tag{1.4}
\end{equation*}
$$

- Alternating polynomials:

$$
\begin{equation*}
\mathbb{C}[x]^{\mathfrak{S}_{n}, \operatorname{sgn}}=\left\{f \in \mathbb{C}[x] \mid \sigma(f)=\operatorname{sgn}(\sigma) f \quad\left(\sigma \in \mathfrak{S}_{n}\right)\right\}=\Delta(x) \mathbb{C}[x]^{\mathfrak{S}_{n}}, \tag{1.5}
\end{equation*}
$$

where $\Delta(x)$ stands for the difference product (Vandermonde determinant):

$$
\begin{equation*}
\Delta(x)=\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)=\operatorname{det}\left(x_{i}^{n-j}\right)_{i, j=1}^{n} \tag{1.6}
\end{equation*}
$$

### 1.2 Monomial symmetric functions

$$
\begin{equation*}
\mathbb{C}[x]^{\mathfrak{G}_{n}}=\bigoplus_{\lambda \in \mathcal{P}_{n}} \mathbb{C} m_{\lambda}(x)=\bigoplus_{\lambda \in \mathcal{P}_{n}} \mathbb{C} s_{\lambda}(x) \tag{1.7}
\end{equation*}
$$

- Each symmetric polynomial $f(x) \in \mathbb{C}[x]^{\mathfrak{G}_{n}}$ is uniquely expressed as a finite linear combination of monomial symmetric functions $m_{\lambda}(x)$.

$$
\begin{equation*}
f(x)=\sum_{\lambda \in \mathcal{P}_{n}} a_{\lambda} m_{\lambda}(x) \quad \text { (finite sum) } \tag{1.8}
\end{equation*}
$$

### 1.3 Schur functions

- The Schur function $s_{\lambda}(x)$ attached to a partition $\lambda \in \mathcal{P}_{n}$ can be defined in two ways.

First definition: (combinatorial definition)

$$
\begin{equation*}
s_{\lambda}^{\mathrm{comb}}(x)=\sum_{T \in \operatorname{SSTab}_{n}(\lambda)} x^{\mathrm{wt}(T)}, \tag{1.9}
\end{equation*}
$$

as a sum of certain weights over all semi-standard tableaux (column strict tableaux) $T$ of shape $\lambda$.

## Second definition:

$$
\begin{equation*}
s_{\lambda}(x)=\frac{\operatorname{det}\left(x_{i}^{\lambda_{j}+n-j}\right)_{i, j=1}^{n}}{\operatorname{det}\left(x_{i}^{n-j}\right)_{i, j=1}^{n}}=\frac{\operatorname{det}\left(x_{i}^{\lambda_{j}+n-j}\right)_{i, j=1}^{n}}{\Delta(x)} \tag{1.10}
\end{equation*}
$$

as a ratio of two determinants of Vandermonde type.
I will explain in the lecture how one can prove the equivalence of two definitions of the Schur functions, on the basis of Cauchy's lemma and the generating function of Cauchy type.

### 1.4 Cauchy's lemma

- Cauchy's lemma: For two sets of varialbes $x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right)$,

$$
\begin{equation*}
\operatorname{det}\left(\frac{1}{1-x_{i} y_{j}}\right)_{i, j=1}^{n}=\frac{\Delta(x) \Delta(y)}{\prod_{i, j=1}^{n}\left(1-x_{i} y_{j}\right)} \tag{1.11}
\end{equation*}
$$

- Generating function: (Kernel function of Cauchy type)

$$
\begin{equation*}
\prod_{i, j=1}^{n} \frac{1}{1-x_{i} y_{j}}=\sum_{\lambda \in \mathcal{P}_{n}} s_{\lambda}(x) s_{\lambda}(y) \tag{1.12}
\end{equation*}
$$

I also give comments on some consequences of the equivalence of the two definitions.

### 1.5 Equivalence of the two definitions

### 1.6 Some remarks on the Schur functions

- Finite dimensional representations of $\mathrm{GL}_{n}$.
- Schur functions in the context of KP hierarchy.

