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# « Introduction to Macdonald polynomials »

2021/02/05.

§ 0: Introduction.

1° Monographs by Macdonald

[1] Symmetric Functions and Hall Polynomials, 2nd Ed.  
Oxford, 1995. 475pp ← Chapter VI

[2] Symmetric Functions and Orthogonal Polynomials  
University Lecture Series 12, AMS, 1998 53pp

[3] Affine Hecke Algebras and Orthogonal Polynomials.  
Cambridge, 2003, 175pp ← Macdonald-Cherednik theory

2° Macdonald polynomials (of  $A_{n-1}$  type,  $GL_n$  version)

$$P_{\lambda}(x) = P_{\lambda}(x; q, t) \in \underline{\mathbb{C}[x]}^{\mathfrak{S}_n} \quad x = (x_1, \dots, x_n)$$

$\mathfrak{S}_n$ : symmetric group  
of degree  $n$

$$P_n \left\{ \begin{array}{l} \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \quad (\mathbb{N} = \mathbb{Z}_{\geq 0}); \\ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0 \quad l(\lambda) \leq n \end{array} \right\}$$

$$\cdot \mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in P_n} \mathbb{C} P_{\lambda}(x) \quad \perp$$

$$\cdot \mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in P_n} \mathbb{C} m_{\lambda}(x) = \bigoplus_{\lambda \in P_n} \mathbb{C} s_{\lambda}(x)$$

$$m_\lambda(x) = \sum_{\mu \in \mathbb{S}_n \lambda} x^\mu \quad : \text{ monomial symmetric functions of monomial type } \lambda$$

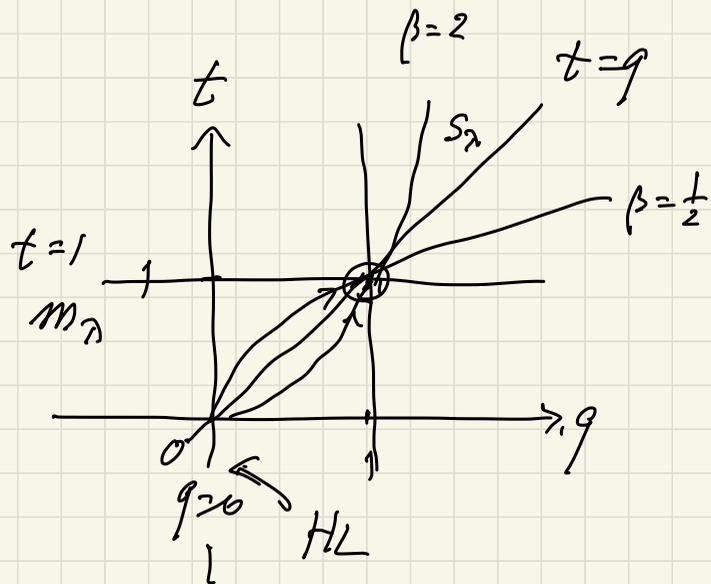
$$= \underline{x^\lambda} + \dots$$

$$S_\lambda(x) = \frac{\det(x_i^{\lambda_j + n - j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\det(\dots)}{\Delta(x)} \quad \Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

$$= \underline{x^\lambda} + \dots \quad \text{Schur functions}$$

... = lower order terms w.r.t. the dominance order.

$$P_\lambda(x; q, t) \text{ specialize to } \begin{cases} m_\lambda(x) & t=1 \\ S_\lambda(x) & t=q \end{cases}$$



- $t = q^\beta, q \rightarrow 1$   $d = \frac{1}{\beta}$   
 $P_\lambda(x; q, q^\beta) \rightarrow P_\lambda^{(\beta)}(x)$   
 Jack polynomials

- $P_\lambda(x; t) = P_\lambda(x; 0, t) \quad q=0$   
 Hall-Littlewood

- $P_\lambda(x; q, 0) \quad t=0$   
 $\infty$   $\infty$   
 $q$ -Whittaker functions

# Macdonald-Ruijsenaars $q$ -difference operator

$$D_x = \sum_{z=1}^n \prod_{\substack{1 \leq j \leq n \\ j \neq z}} \frac{t x_i - x_j}{x_i - x_j} P_{q, x_i} = \prod_{j=2}^n \frac{t x_1 - x_j}{x_1 - x_j} P_{q, x_1} + \dots$$

$$P_{q, x_i} f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, q x_i, \dots, x_n) \quad (i=1, \dots, n)$$

$q$ -shift operator in  $x_i$

$$D_x : \mathbb{C}[x]^{\mathfrak{S}_n} \longrightarrow \mathbb{C}[x]^{\mathfrak{S}_n} \quad \overline{\overline{\Delta(x) D_x}}$$

# Theorem (Macdonald)

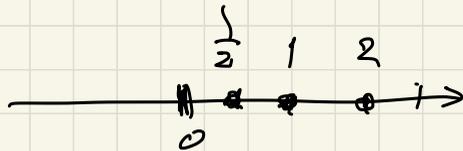
$$\forall \lambda \in \mathcal{P}_n; \exists! P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{Q}(q, t)[x] \quad \mathbb{S}_n$$

$$(1) \quad D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad (d_\lambda = \sum_{i=1}^n t^{n-i} q^{d_i})$$

$$(2) \quad P_\lambda(x) = x^\lambda + (\text{lower order terms}) \\ = \underline{\underline{m_\lambda(x)}} + (\text{lower order terms})$$

# Remark

- Special values  $\beta$ ,  $t = q^\beta$



Jack polynomials  $P_\lambda^{(\beta)}(x)$

— zonal spherical functions associated  
with symmetric pairs

*coideal  
subalgebra*

- $\beta = \frac{1}{2}$ :  $GL_n / SO_n$  zonal polynomials
- $\beta = 1$ :  $GL_n \times GL_n / GL_n$   $S_\lambda = \text{character of } GL_n$
- $\beta = 2$ :  $GL_{2n} / Sp_{2n}$  — quantum symmetric pairs

Macdonald polynomials  $P_\lambda(x; q, t) := (U_q(gl_N), U_q(\mathfrak{k}))$

- General values of  $\beta$ ,

$P_{\lambda}^{(p)}(x)$  : (orthogonal polynomials  
joint eigen functions)

non-relativistic

- Heckman - Opdam system of differential equation

→ Calogero - Sutherland system (model)

$P_{\lambda}(x; q, t)$  • Macdonald - Ruijsenaars system of  $q$ -difference equation

$q$ -difference version of Calogero Sutherland model  
relativistic

3<sup>o</sup>: Fundamental properties.

(a) Specializations: Schur, Jack, H-L,  $q$ W.

(b)  $q$ -Difference equations:  $r=1, 2, \dots, n$

$$D_x^{(r)} = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=r}} t^{\binom{r}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q, x_i}$$

$$[D_x^{(r)}, D_x^{(s)}] = 0$$

$$t^{\delta} q^{\lambda} = (t^{n-1} q^{\lambda_1}, t^{n-2} q^{\lambda_2}, \dots, q^{\lambda_n})$$

$$D_x^{(r)} P_{\lambda}(x) = e_r(t^{\delta} q^{\lambda}) P_{\lambda}(x)$$

$$\delta = (n-1, n-2, \dots, 0)$$

$e_r$ :  $r$ th elementary symmetric function

staircase 

### (c) orthogonality

$$w(x; q, t) = \prod_{1 \leq i < j \leq n} \frac{(x_i/x_j; q)_{\infty} (t x_j/x_i; q)_{\infty}}{(t x_i/x_j; q)_{\infty} (t x_j/x_i; q)_{\infty}} \leftarrow$$

$$(z; q)_{\infty} = \prod_{i=0}^{\infty} (1 - q^i z) \quad (|q| < 1)$$

$$\langle f(x), g(x) \rangle = \text{const} \int_{\mathbb{R}^n} f(x) g(x) w(x; q, t) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$

$$|x_1| = \dots = |x_n| = 1$$

*constant term conjecture  
scalar product conjecture*

$$\left( w^{(\beta)}(x) = \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\beta} \right)$$

$$\left\langle \frac{P_{\lambda}(x), P_{\lambda}(x)}{\langle 1, 1 \rangle} \right\rangle ?$$

(d) Evaluation formula and self-duality

$$\left\{ \begin{array}{l} t^\delta = (t^{\delta_1}, t^{\delta_2}, \dots, 1) \text{ base point} \end{array} \right.$$

$$\left\{ \begin{array}{l} P_\lambda(t^\delta) = \dots \text{ explicit formula} \end{array} \right.$$

$$\tilde{P}_\lambda(x) = \frac{P_\lambda(x)}{P_\lambda(t^\delta)} \quad : \quad \tilde{P}_\lambda(t^\delta) = 1$$

Symmetry

$$\tilde{P}_\lambda(\underline{t^\delta q^\mu}) = \tilde{P}_\mu(\underline{t^\delta q^\lambda}) \quad \lambda, \mu \in P_n$$

$x = t^\delta q^\mu$   
position variable

$\xi = t^\delta q^\lambda$   
spectral variable.

# § 1: Schur functions

1° Symmetric polynomials  $e_r(x)$ ,  $h_\ell(x)$ ,  $p_k(x)$

$$e_r(x) = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \dots x_{i_r} \quad \text{elementary symmetric polynomial} \\ (r=0, 1, 2, \dots)$$

$$h_\ell(x) = \sum_{\mu_1 + \dots + \mu_n = \ell} x_1^{\mu_1} \dots x_n^{\mu_n} = \sum_{1 \leq j_1 \leq \dots \leq j_\ell \leq n} x_{j_1} \dots x_{j_\ell} \quad (\ell=0, 1, 2, \dots)$$

complete (homogeneous) symmetric polynomial

$$p_k(x) = x_1^k + \dots + x_n^k \quad (k=1, 2, \dots)$$

# Facts

(1) Symmetric polynomials

$$\mathbb{C}[x]^{\mathbb{S}_n} = \{ f \in \mathbb{C}[x] \mid \sigma(f) = f \quad (\sigma \in \mathbb{S}_n) \}$$
$$= \mathbb{C}[e_1(x), \dots, e_n(x)] \quad \leftarrow$$

$e_1(x), \dots, e_n(x)$ : algebraically independent

(2) Alternating polynomials

$$\mathbb{C}[x]^{\mathbb{S}_n, \text{sgn}} = \{ f \in \mathbb{C}[x] \mid \sigma(f) = \text{sgn}(\sigma) f \quad (\sigma \in \mathbb{S}_n) \}$$
$$= \underline{\underline{\Delta(x)}} \mathbb{C}[x]^{\mathbb{S}_n}.$$

$h_2(x), p_2(x), \dots$

④ How can one express  $h_0(x), p_0(x)$  in terms of  $e_k$ ?

$$E(u) = 1 - e_1(x)u + e_2(x)u^2 - \dots + (-1)^n e_n(x)u^n \quad e_0(x) = 1$$

$$= (1 - x_1 u)(1 - x_2 u) \dots (1 - x_n u)$$

$$H(u) = 1 + h_1(x)u + h_2(x)u^2 + \dots$$

$$h_0(x) = 1$$

$$= \frac{1}{(1 - x_1 u)(1 - x_2 u) \dots (1 - x_n u)}$$



$$(1 + x_1 u + x_1^2 u^2 + \dots) (1 + x_2 u + x_2^2 u^2 + \dots) \dots (1 + x_n u + x_n^2 u^2 + \dots)$$

$$H(u)E(u) = 1$$

$$\sum_{\ell+r=k} (-1)^r h_\ell(x) e_r(x) = \begin{cases} 1 & (k=0) \\ 0 & (k>0) \end{cases}$$

 $u^k$ 

Wronski relation

$$1 = 1$$

$$\underline{e_1} - h_1 = 0$$

$$h_1 = e_1$$

$$e_2 - h_1 e_1 + h_2 = 0$$

$$h_2^2 - e_2 + h_1 e_1 = -e_2 + e_1^2 \dots$$

 $\vdots$ 

$$e_n - h_1 e_{n-1} + \dots + (-1)^n h_n = 0$$

