

Introduction to Macdonald polynomials: Lecture 2

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1 Schur functions

1.1 Symmetric polynomials $e_r(x)$, $h_l(x)$ and $p_k(x)$

- Three sequences of symmetric polynomials in $x = (x_1, x_2, \dots, x_n)$

$e_r(x)$: elementary symmetric functions, $h_l(x)$: complete homogeneous symmetric functions

$$\begin{aligned} e_r &= e_r(x) = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r} \quad (r = 0, 1, 2, \dots), \quad e_r(x) = 0 \quad (r > n), \\ h_l &= h_l(x) = \sum_{\mu_1 + \dots + \mu_n = l} x_1^{\mu_1} \cdots x_n^{\mu_n} = \sum_{1 \leq j_1 \leq \dots \leq j_l \leq n} x_{j_1} x_{j_2} \cdots x_{j_l} \quad (l = 0, 1, 2, \dots). \end{aligned} \quad (1.1)$$

$p_k(x)$: power sums

$$p_k = p_k(x) = x_1^k + x_2^k + \cdots + x_n^k \quad (k = 1, 2, \dots). \quad (1.2)$$

- The ring of symmetric polynomials:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \{f \in \mathbb{C}[x] \mid \sigma(f) = f \quad (\sigma \in \mathfrak{S}_n)\} = \mathbb{C}[e_1(x), \dots, e_n(x)]. \quad (1.3)$$

The elementary symmetric polynomials $e_1(x), \dots, e_n(x)$ are algebraically independent over \mathbb{C} .

- Alternating polynomials:

$$\mathbb{C}[x]^{\mathfrak{S}_n, \text{sgn}} = \{f \in \mathbb{C}[x] \mid \sigma(f) = \text{sgn}(\sigma)f \quad (\sigma \in \mathfrak{S}_n)\} = \Delta(x)\mathbb{C}[x]^{\mathfrak{S}_n}, \quad (1.4)$$

where $\Delta(x)$ stands for the difference product (Vandermonde determinant):

$$\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det (x_i^{n-j})_{i,j=1}^n \quad (1.5)$$

- How can one express $h_l(x)$ and $p_k(x)$ in terms of $e_k(x)$?

$$\begin{aligned} h_1 &= e_1, \quad h_2 = e_1^2 - e_2, \quad h_3 = e_1^3 - 2e_1e_2 + e_3, \quad h_4 = e_1^4 - 3e_1^2e_2 + 2e_1e_3 + e_2^2 - e_4, \dots \\ p_1 &= e_1, \quad p_2 = e_1^2 - 2e_2, \quad p_3 = e_1^3 - 3e_1e_2 + e_3, \quad p_4 = e_1^4 - 4e_1^2e_2 + 4e_1e_3 + 2e_2^2 - 4e_4, \dots \end{aligned} \quad (1.6)$$

- Wronski formulas: $h_l \longleftrightarrow e_r$.

$$h_l - h_{l-1}e_1 + h_{l-2}e_2 - \cdots + (-1)^{l-1}h_1e_{l-1} + (-1)^l e_l = 0 \quad (l = 1, 2, \dots)$$

$$h_l = \det \begin{bmatrix} e_1 & 1 & & & \\ e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ e_{l-1} & e_{l-2} & \dots & e_1 & 1 \\ e_l & e_{l-1} & \dots & e_2 & e_1 \end{bmatrix}, \quad h_l = \sum_{\|m\|=l} \frac{(-1)^{l-|m|}|m|!}{m_1!m_2!\cdots m_n!} e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n}, \quad (1.7)$$

where $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, $|m| = m_1 + m_2 + \dots + m_n$, $\|m\| = m_1 + 2m_2 + \dots + nm_n$.

- **Newton formulas:** $p_k \longleftrightarrow e_r$.

$$p_k - p_{k-1}e_1 + p_{k-2}e_2 - \dots + (-1)^{k-1}p_1e_{k-1} + (-1)^kke_k = 0 \quad (k = 1, 2, \dots)$$

$$p_k = \det \begin{bmatrix} e_1 & 1 & & & \\ 2e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (k-1)e_{k-1} & e_{k-2} & \dots & e_1 & 1 \\ ke_k & e_{k-1} & \dots & e_2 & e_1 \end{bmatrix}, \quad \frac{p_k}{k} = \sum_{\|m\|=k} \frac{(-1)^{k-|m|}(|m|-1)!}{m_1! \cdots m_n!} e_1^{m_1} \cdots e_n^{m_n}, \quad (1.8)$$

- **Generating functions:** These recurrence formulas, determinant formulas and explicit formulas can be proved by means of the generating functions.

$$\begin{aligned} E(u) &= 1 - e_1u + e_2u^2 - \dots + (-1)^n e_n u^n = (1 - x_1u) \cdots (1 - x_nu) \quad (e_0 = 1), \\ H(u) &= 1 + h_1u + h_2u^2 + h_3u^3 + \dots = \frac{1}{(1 - x_1u) \cdots (1 - x_nu)} \quad (h_0 = 1). \end{aligned} \quad (1.9)$$

The Wronski formulas arise from the relation $E(u)H(u) = 1$ of the generating functions. As to power sums, we have

$$H(u) = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} p_k u^k \right), \quad E(u) = \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} p_k u^k \right), \quad (1.10)$$

The Newton formulas are obtained from

$$P(u) = \frac{uH'(u)}{H(u)} = -\frac{uE'(u)}{E(u)}, \quad P(u) = \sum_{k=1}^{\infty} p_k u^k. \quad (1.11)$$

1.2 Monomial symmetric functions

- **Two \mathbb{C} -bases of $\mathbb{C}[x]^{\mathfrak{S}_n}$**

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}m_{\lambda}(x) = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}s_{\lambda}(x) \quad (1.12)$$

$m_{\lambda}(x)$: monomial symmetric function of monomial type λ :

$$m_{\lambda}(x) = \sum_{\mu \in \mathfrak{S}_n \cdot \lambda} x^{\mu} = x^{\lambda} + (\text{monomials obtained from } x^{\lambda} \text{ by permutations}) \quad (1.13)$$

Note that $\lambda \in \mathcal{P}_n$, $\mu \in \mathfrak{S}_n \cdot \lambda \implies \mu \leq \lambda$ w.r.t. the *dominance order* (partial order).

Single column: $m_{(1^r)}(x) = e_r(x)$. Single row: $m_{(l)}(x) = p_l(x)$,

- Each symmetric polynomial $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ is uniquely expressed as a finite linear combination of monomial symmetric functions $m_{\lambda}(x)$.

$$f(x) = \sum_{\lambda \in \mathcal{P}_n} a_{\lambda} m_{\lambda}(x) \quad (\text{finite sum}) \quad (1.14)$$

1.3 Schur functions

- The Schur functions $s_\lambda(x)$ attached to a partition $\lambda \in \mathcal{P}_n$ can be defined in two ways.

First definition: (combinatorial definition)

$$s_\lambda^{\text{comb}}(x) = \sum_{T \in \text{SSTab}_n(\lambda)} x^{\text{wt}(T)}, \quad (1.15)$$

as a sum of certain weights over all *semi-standard tableaux* (column strict tableaux) T of shape λ .

Second definition: (determinantal definition)

$$s_\lambda^{\det}(x) = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\Delta(x)} \quad (1.16)$$

as a ratio of two determinants of Vandermonde type.

Theorem $s_\lambda^{\text{comb}}(x) = s_\lambda^{\det}(x) \quad (\lambda \in \mathcal{P}_n).$

This theorem implies that $s_\lambda(x) \in \mathbb{N}[x]^{\mathfrak{S}_n}$, $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

We will explain below how one can prove the equivalence of two definitions of the Schur functions, on the basis of Cauchy's lemma and the generating function of Cauchy type. At this moment, we explain some consequences of this theorem.

- **Evaluation formula:** The value of $s_\lambda(x)$ at the point $t^\delta = (t^{n-1}, t^{n-2}, \dots, 1)$ is evaluated as

$$s_\lambda(t^\delta) = t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \frac{1 - t^{\lambda_i - \lambda_j + j - i}}{1 - t^{j-i}}, \quad n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i. \quad (1.17)$$

From this, we obtain the *hook-length formula* for $s_\lambda(1, \dots, 1) = \#\text{SSTab}_n(\lambda)$:

$$s_\lambda(1, \dots, 1) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \quad \#\text{SSTab}_n(\lambda) = \prod_{s \in \lambda} \frac{n + c_\lambda(s)}{h_\lambda(s)} \quad (1.18)$$

For each box $s = (i, j) \in \lambda$, the *content* $c_\lambda(s)$ and the *hook-length* $h_\lambda(s)$ are defined by

$$c_\lambda(s) = j - i, \quad h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1 = \lambda_i - j + \lambda'_j - i + 1. \quad (1.19)$$

- **Self-duality:**

$$\frac{s_\lambda(t^{\mu+\delta})}{s_\lambda(t^\delta)} = \frac{s_\mu(t^{\lambda+\delta})}{s_\mu(t^\delta)} \quad (\lambda, \mu \in \mathcal{P}_n). \quad (1.20)$$

1.4 Cauchy's lemma

In what follows, we set $s_\lambda(x) = s_\lambda^{\det}(x)$.

- **Cauchy's lemma:** For two sets of variables $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$,

$$\det \left(\frac{1}{1 - x_i y_j} \right)_{i,j=1}^n = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^n (1 - x_i y_j)}. \quad (1.21)$$

- **Generating function:** (Kernel function of Cauchy type)

$$\prod_{i,j=1}^n \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}_n} s_\lambda(x)s_\lambda(y). \quad (1.22)$$

1.5 Equivalence of the two definitions

For a semi-standard tableau T of shape λ , consider the subdiagram μ consisting of the boxes $s \in \lambda$ such that $T(s) \leq n-1$. Then μ is a partition, and the skew diagram (difference) λ/μ is a *horizontal strip*, i.e. $\mu \subseteq \lambda$ satisfies the condition $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ ($i \geq 1$). The combinatorial definition of $s_\lambda^{\text{comb}}(x)$ can be understood as the recursion formula with respect to n :

$$s_\lambda^{\text{comb}}(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda: \text{ h.s.}} s_\mu^{\text{comb}}(x_1, \dots, x_{n-1}) x_n^{|\lambda| - |\mu|}. \quad (1.23)$$

In order to establish the equivalence $s_\lambda^{\text{comb}}(x) = s_\lambda^{\text{det}}(x)$, we prove that $s_\lambda(x) = s_\lambda^{\text{det}}(x)$ satisfies the same recursion formula by using Cauchy's lemma.

1.6 Some remarks on the Schur functions