Lecture 2 : 2021/02/12

§1: Schur functions

1° Symmetric polynomials e_r, h_r, p_r

Exercise : Prove (Confirm) these formulas by yourself

2° Monomial symmetric functions

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x) = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, s_\lambda(x)$$

$\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$

$\vdots$

$x = (x_1, \dots, x_n)$

$\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$: ring of polynomials in x

$$f = f(x) = f(x_1, \dots, x_n) = \sum_{\mu_1, \dots, \mu_n \geq 0} a_{\mu_1 \dots \mu_n} x_1^{\mu_1} \cdots x_n^{\mu_n} \text{ (finite)}$$

- <u>notation of multi-indices</u> <u>composition</u>

$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$. $\mathbb{N} = \mathbb{Z}_{\geq 0}$

$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ $\quad \deg(x^\mu) = \mu_1 + \dots + \mu_n = |\mu|$

- $\mathfrak{S}_n = \{ \sigma : \{1, \dots, n\} \to \{1 \dots n\} \mid \text{bijection} \}$ symmetric group of degree n

$f = f(x_1, \dots, x_n)$

- $\mathfrak{S}_n \curvearrowright \mathbb{C}[x]$ $\quad \sigma \in \mathfrak{S}_n$

$$\sigma(f) = f(\boxed{x_{\sigma(1)}, \dots, x_{\sigma(n)}}) = \sum_{\mu_1 \dots \mu_n} a_{\mu_1 \dots \mu_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}$$

τ permutation of variables $\quad x_i \to x_{\sigma(i)}$

$\sigma^{-1}.(a_1 \dots a_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$

$$\sigma(x^\mu) = x_{\sigma(1)}^{\mu_1} \cdots \underbrace{x_{\sigma(i)}}_{j}^{\mu_i} \cdots x_{\sigma(n)}^{\mu_n} = x_1^{\mu_{\sigma^{-1}(1)}} \cdots x_j^{\mu_{\sigma^{-1}(j)}} \cdots x_n^{\mu_{\sigma^{-1}(n)}}$$

$i = \sigma^{-1}(j)$

- $\mathfrak{S}_n \curvearrowright \mathbb{N}^n$ $\quad \sigma \in \mathfrak{S}_n \quad \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$

$$\sigma.\mu = (\mu_{\sigma^{-1}(1)}, \dots, \mu_{\sigma^{-1}(n)})$$

left action

$$\sigma(x^\mu) = x^{\sigma.\mu}$$

σ permute the positions

$\sigma(f)(a) = f(\sigma^{-1}.a)$

<u>Example</u> $n=3$, $\sigma = (1\,2\,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

cyclic perm

$$\mu = (\overset{1}{\mu_1}, \overset{2}{\mu_2}, \overset{3}{\mu_3})$$

σ

$$\sigma.\mu = (\underset{1}{\mu_3}, \underset{2}{\mu_1}, \underset{3}{\mu_2})$$

$\mu_{\sigma^{-1}(1)}$

$$\underline{\underline{\sigma.\mu = (\mu_{\sigma^{-1}(1)}, \dots, \mu_{\sigma^{-1}(n)})}}$$

- $$f = \sum_{\mu \in \mathbb{N}^n} a_\mu x^\mu \quad \text{(finite sum)}$$

$$\sum_\mu a_{\sigma^{-1}.\mu} x^\mu$$

$$\sigma(f) = \sum_{\mu \in \mathbb{N}^n} a_\mu \sigma(x^\mu) = \sum_\mu a_\mu x^{\overbrace{\sigma.\mu}^{=\nu}} = \sum_\nu a_{\sigma^{-1}.\nu} x^\nu$$

$$\sigma(f) = f \iff a_{\mu} = a_{\sigma^{-1}\cdot\mu} \quad (\mu \in \mathbb{N}^n)$$

$\mathfrak{S}_n$

$f \in \mathbb{C}[x]^{\mathfrak{S}_n} \iff$ Regarded as a function of $\mu \in \mathbb{N}^n$, a_μ is constant on each $\mathfrak{S}_n$-orbit.

$$m_\lambda(x) = \sum_{\mu \in \mathfrak{S}_n \cdot \lambda} x^\mu$$

monomial symmetric function (orbit sum)

$$\mathfrak{S}_n \backslash \mathbb{N}^n = \{ \mathfrak{S}_n \cdot \mu \mid \mu \in \mathbb{N}^n \}$$

$$\mathcal{P}_n = \{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0 \}$$

partition λ with $\ell(\lambda) \leq n$

$$\mathfrak{S}_n \cdot \mu \cap \mathcal{P}_n = \{\lambda\} \qquad \lambda = \mu^+ \qquad \mathcal{P}_n \xrightarrow{\sim} \mathfrak{S}_n \backslash \mathbb{N}^n$$

$$\mathbb{C}[x]^{S_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x)$$

$$f(x) = \sum_{\lambda \in \mathcal{P}_n} a_\lambda\, m_\lambda(x) \quad (\text{finite sum})$$

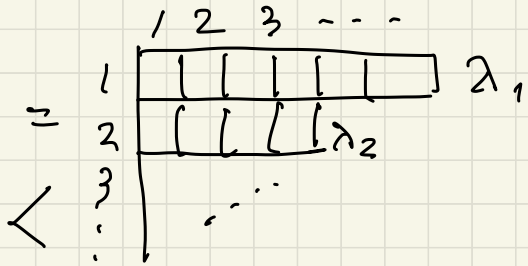
<u>partition</u>

$\lambda = (\lambda_1, \lambda_2, \ldots, 0, \ldots)$ $\qquad$ $\lambda_i \in \mathbb{N}$ $(i = 1, 2, \ldots)$

$\lambda_i = 0$ $(i \gg 0)$

$\ell(\lambda) = \#$ of positive parts $\qquad$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > \lambda_{\ell+1} = 0 \cdots$

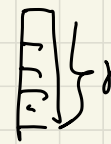
$\ell(\lambda) = 0 \qquad \lambda = (0, 0, \ldots) = () = \phi$

$\lambda =$ 

Young diagram

$$\lambda \overset{\text{def}}{=} P(\lambda) = \left\{ s = (i,j) \;\middle|\; \begin{array}{l} i \geq 1 \\ 1 \leq j \leq \lambda_i \end{array} \right\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$$

<u>Example</u> $\lambda \in \mathcal{P}_n \quad \ell(\lambda) \leq n$

(1) $\lambda = (\overbrace{1 \cdots 1}^{r} 0 \cdots 0) = (1^r) =$  single column

$x_1 \cdots x_r$

$$m_{(1^r)}(x) = x_1 \cdots x_r + \text{terms obtained by permutation}$$

$$= \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} = e_r(x)$$

(2) $\lambda = (\ell, 0, 0 \ldots) = (\ell) = \overbrace{\square\square\square\square\square\square\square}^{\ell}$ single row

$$m_{\square\square\square\square}(x) = x_1^{\ell} + x_2^{\ell} + \cdots + x_n^{\ell} = p_{\ell}(x)$$

(3) $n = 3$, $\lambda = (2,1,0) = \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ $\quad |\mathfrak{S}_3| = 6.$

deg $= 3$

$$m_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}(x) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

6 terms

$$m_{\square\square\square}(x) = x_1^3 + x_2^3 + x_3^3$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$m_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}}(x) = x_1 x_2 x_3$$

$\lambda \in \mathcal{P}_n$

$$m_{\lambda}(x) = x^{\lambda} + (\text{lower order terms})$$

- <u>dominance order</u> (partial order) on $\mathbb{N}^n$

$\mu = (\mu_1, \dots, \mu_n),\ \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$

$$\mu \leq \nu \iff \begin{cases} \mu_1 \leq \nu_1 \\ \mu_1 + \mu_2 \leq \nu_1 + \nu_2 \\ \vdots \\ \mu_1 + \dots + \mu_{n-1} \leq \nu_1 + \dots + \nu_{n-1} \\ \mu_1 + \dots + \mu_n = \nu_1 + \dots + \nu_n \end{cases}$$

$\boxed{A_{n-1}}$

$(|\mu| = |\nu|)$

<u>Exercise</u> $\lambda \in \mathcal{P}_n$: $\mu \in \mathfrak{S}_n \cdot \lambda \Longrightarrow \mu \leq \lambda$

<u>Exercise</u> $\mu, \nu \in \mathbb{N}^n$, $\mu \leq \nu \Longrightarrow \mu \underset{\text{lex}}{\leq} \nu$

lexicographic order

3° Schur functions

$$\mathbb{C}[x]^{S_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, s_\lambda(x)$$

$$s_\lambda(x) = \underline{x}^\lambda + (\text{lower order terms})$$

$$= m_\lambda(x) + (\quad " \quad)$$

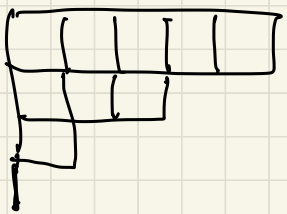
$$\underline{\underline{s_\lambda^{comb}(x)}} = \underline{\underline{s_\lambda^{det}(x)}}$$

(11) First definition (combinatorial def)

$$s_\lambda(x) = \sum_{T \in SSTab_n(\lambda)} x^{wt(T)}$$

column strict tableaux

- $SSTab_n(\lambda)$ = the set of semi-standard tableaux of shape λ $\{1, \dots, n\}$

$\lambda =$ 

$\lambda = (5, 3, 1, 0)$ $n = 4$

$T =$

1	1	2	3	3
2	3	4		
4				

semi-stand

$a \le b$

$\wedge$

c

standard

$\left(\begin{array}{c} a < b \\ \wedge \\ c \end{array} \right)$

$1 \sim n$

$$T : D(\lambda) \longrightarrow \{1, \dots, n\} \qquad \begin{cases} T(i,j) \le T(i,j+1) \\ T(i,j) < T(i+1,j) \end{cases}$$

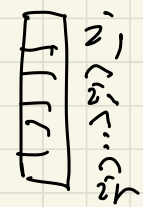
$$\mathrm{wt}(T) = \underset{\in \mathbb{N}^n}{\mu} = (\mu_1, \dots, \mu_n) \qquad \mu_i = \#\text{ of } i\text{'s in } T \quad (i = 1, \dots, n)$$

$$x^{\mathrm{wt}(T)} = x^{\mu} = x_1^{\mu_1} \cdots x_n^{\mu_n}$$

Examples

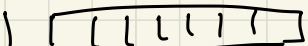
(1) $\lambda = (1^r)$

$$s^{\text{comb}}_{\square\,\square\,\square\,\square}(x) = \sum_{1 \le i_1 < \dots < i_r \le n} x_{i_1} \cdots x_{i_r} = e_r(x)$$



(2) $\lambda = (\ell)$

$$s^{\text{comb}}_{\square\,\square\,\square\,\square\,\square\,\square}(x) = \sum_{1 \le j_1 \le j_2 \le \dots \le j_\ell \le n} x_{j_1} \cdots x_{j_\ell} = h_\ell(x)$$



$j_1 \le j_2 \le \dots \le j_\ell$

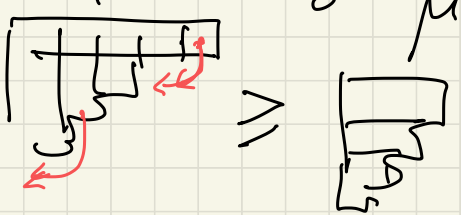
Example $n=3$, $\lambda=(2,1,0)=$ $\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}$ **SSTab$_n(\lambda)$?**

1 1	1 1	1 2	1 2
2	3	2	3
$x_1^2 x_2$	$x_1^2 x_3$	$x_1 x_2^2$	$x_1 x_2 x_3$

1 3	1 3	2 2	2 3
2	3	3	3
$x_1 x_2 x_3$	$x_1 x_3^2$	$x_1^2 x_3$	$x_2 x_3^2$

$$s_{\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}}(x_1,x_2,x_3)$$
$$= \left(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_1^2x_3 + x_2x_3^2\right)$$
$$+\, 2x_1x_2x_3$$
$$= m_{\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}}(x) + 2\, m_{\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}}(x)$$

λ $\quad$ μ



$\geq$

$\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array} \geq \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$

$(2,1,0)$ $\quad$ $(1,1,1)$

$s_\lambda^{comb}(x) \in \mathbb{N}[x]$ but why $\in \mathbb{N}[x]^{\mathfrak{S}_n}$?

- Kostka numbers $\lambda \vdash \mathcal{P}_n$, $\mu \in \mathbb{N}^n$

$$SSTab_n(\lambda)_\mu = \{ T \in SSTab_n(\lambda) \mid wt(T) = \mu \}$$

$K_{\lambda\mu} = \# SSTab_n(\lambda)_\mu$: Kostka number

$$s_\lambda^{comb}(x) = \sum_{T \in SSTab_n(\lambda)} x^{wt(T) = \mu}$$

$$= \sum_{\mu \in \mathbb{N}^n} \underbrace{\left(\# SSTab_n(\lambda)_\mu \right)}_{K_{\lambda\mu}} x^\mu$$

$$s_\lambda^{comb}(x) = \sum_{\mu \in \mathbb{N}^n} K_{\lambda\mu} \underline{\underline{x^\mu}}$$

$$= x^\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} x^\mu$$

<u>Exercise</u> $K_{\lambda\lambda} = 1$, $K_{\lambda\mu} > 0 \Rightarrow \mu \leq \lambda$ (dominance order)

$$\underbrace{s_\lambda^{comb}(x) \in \mathbb{N}[x]^{\mathfrak{S}_n}}_{\uparrow} \iff K_{\lambda\mu} = K_{\lambda\, \sigma.\mu}$$

$(\mu \in \mathbb{N}^n,\ \sigma \in \mathfrak{S}_n)$

Bender-Knuth involution

(ii) Second definition (determinantal def)

$$s_\lambda^{\det}(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}} = \frac{\det\left(x_i^{\ell_j}\right)_{i,j=1}^{n}}{\underbrace{\det\left(x_i^{\delta_j}\right)_{i,j=1}^{n}}_{\Delta(x)}}$$

$\in \mathbb{Z}[x]^{S_n}$

$\delta = (n-1, n-2, \dots, 0) \leftarrow$

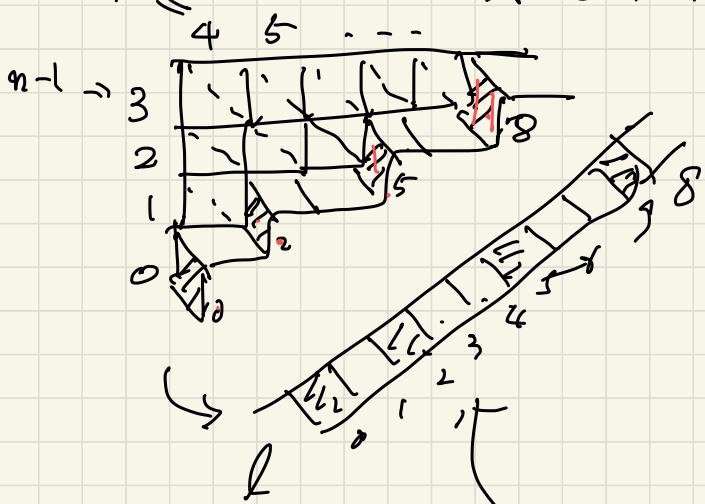
$\lambda = (\lambda_1, \dots, \lambda_n)$

$$\lambda + \delta = (\lambda_1 + n-1, \lambda_2 + n-2, \dots, \lambda_n)$$

$$\begin{array}{ccc} \| & \| & \| \\ \ell & \ell_1 & \ell_n \end{array}$$

but why $\mathbb{N}[x]^{S_n}$?

$\lambda = (5, 3, 1, 0)$ $\quad n = 4$



$\lambda = (5, 3, 1, 0)$

$\delta = (3, 2, 1, 0)$

$l = \lambda + \delta = (8, 5, 2, 0)$

boundary sequence

Maya diagram $\{l_1, \dots, l_n\} \subseteq \mathbb{Z}_{\geq 0} = \mathbb{N}$

Example $n=3$, $\lambda=(2,1,0)$ $\qquad$ $\underline{s_\lambda^{comb}(x)}$

$$s_\lambda^{det}(x) = \frac{\det\begin{pmatrix} x_1^4 & x_1^2 & x_1^0 \\ x_2^4 & x_2^2 & x_2^0 \\ x_3^4 & x_3^2 & x_3^0 \end{pmatrix}}{\Delta(x_1,x_2,x_3)} = \frac{\Delta(x_1^2, x_2^2, x_3^2)}{\Delta(x_1,x_2,x_3)}$$

$\lambda=(2,1,0)$

$\delta=(2,1,0)$

$\ell = \lambda+\delta = (4,2,0)$

$$= \frac{(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}$$

$$= (x_1+x_2)(x_1+x_3)(x_2+x_3) \qquad \underline{\underline{x_1^3}}$$

$$= (\underbrace{x_1^2x_2 + \cdots}) + 2\,\underline{x_1x_2x_3}$$

$$= m_{\overset{(2,1,0)}{\boxplus}}(x) + 2\,m_{\overset{6}{\boxplus}}(x) \quad //$$

Example

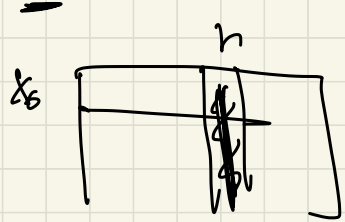
$$s^{\text{def}}_{\boxed{1\,1}\,\ell}(x) = h_\ell(x)$$

(1) $\lambda = (1^r)$ $\quad s^{\text{def}}_{\boxed{1}\,r}(x) = e_r(x)$ $\qquad$ (2): $\underline{\underline{\lambda = (\ell)}}$ row

$$\Delta(x_0, x_1, \dots, x_n) = \prod_{j=1}^{n} (x_0 - x_j)\, \Delta(x_1, \dots, x_n)$$

$$= \left[\sum_{r=0}^{n} (-1)^r x_0^{n-r} \underline{e_r(x)} \right] \Delta(x_1, \dots, x_n)$$

$$\Delta(x_0, x_1, \dots, x_n) = \sum_{r=0}^{n} (-1)^r x_0^{n-r} \det \begin{pmatrix} x_i & \cdots & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$



Theorem $s^{\text{comb}}_\lambda(x) = s^{\text{def}}_\lambda(x)$,

- consequences ...
- Cauchy's Lemma

$$\boxed{s_\lambda^{comb}(x) = s_\lambda^{def}(x)}$$