Lecture 3: 2021/02/19

§1: Schur functions (2)

3° Two definitions of the Schur functions

$$s_\lambda^{comb}(x) = \sum_{T \in SSTab_n(\lambda)} x^{wt(T)}, \qquad s_\lambda^{det}(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\Delta(x)}$$

$$\mathbb{N}[x]^{\mathfrak{S}_n}$$

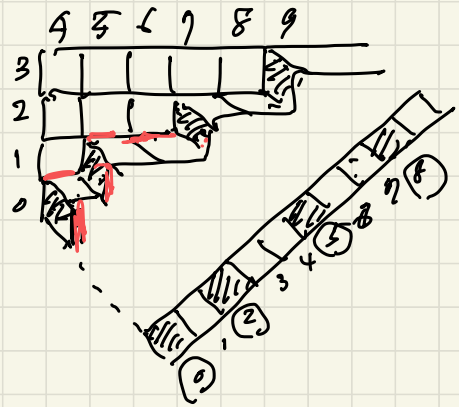
$l_i = \lambda_i + n - i \quad (i = 1, \dots, n)$

$(l_1, \dots, l_n) = (\lambda_1, \dots, \lambda_n) + (n-1, n-2, \dots, 0)$

$$l = \lambda + \delta$$

$n = 4 \qquad \lambda = (5, 3, 1, 0)$

$\lambda =$



$\lambda = (5, 3, 1, 0)$

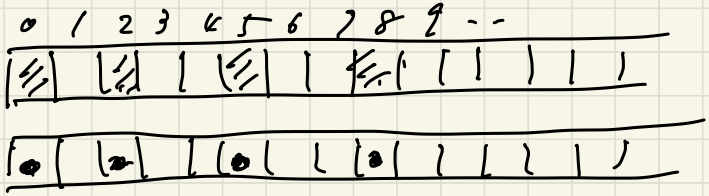
$\delta = (3, 2, 1, 0)$

$\ell = (8, 5, 2, 0)$

boundary sequence

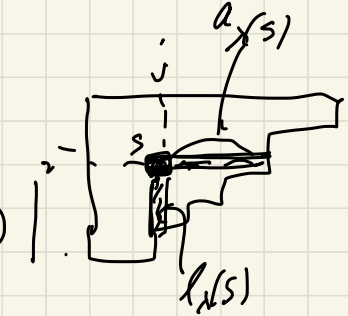
Maya diagram

$M = \{0, 2, 5, 8\} \subseteq \mathbb{N} = \mathbb{Z}_{\geq 0}$



Some consequences

$$s_\lambda(\underline{1,1,\dots,1}) = \sum_{T \in SSTab_n(\lambda)} 1 = |SSTab_n(\lambda)|.$$



$$s_\lambda(x) = \frac{\det(x_i^{\delta_j})_{i,j=1}^n}{\Delta(x)} \qquad \Delta(x) = \det(x_i^{\delta_j})_{i,j=1}^n = \prod_{1 \le i < j \le n}(x_i - x_j)$$

$$\underline{t^\delta = (t^{n-1}, t^{n-2}, \dots, 1)}$$

$$\underline{h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1}$$

Evaluation formula

$$s_\lambda(t^\delta) = t^{n(\lambda)} \prod_{1 \le i < j \le n} \frac{1 - t^{\lambda_i - \lambda_j + j - i}}{1 - t^{j-i}} = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - t^{n + c_\lambda(s)}}{1 - t^{h_\lambda(s)}}$$

$$n(\lambda) = \sum_{j \ge 1} (j-1)\lambda_j$$

$s = (i,j) \in \lambda$

$c_\lambda(s) = j - i$ content

$h_\lambda(s)$ = hook length

$t \to 1$

$$s_\lambda(1,\dots,1) = |SSTab_n(\lambda)| = \prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i} = \prod_{s\in\lambda} \frac{n+c_\lambda(s)}{h_\lambda(s)}$$

hook length formula

Example

$n=3,\ \lambda=(2,1,0)$

$$\frac{n+c_\lambda(s)}{h_\lambda(s)} = \frac{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}} = 2\cdot 4 = 8 :$$

$n=4 \quad \lambda=(5,3,1,0)$

$$\frac{\begin{array}{|c|c|c|c|c|}\hline 4 & 5 & 6 & 7 & 8 \\ \hline 3 & 4 & 5 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|c|c|}\hline 7 & 5 & 4 & 2 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}} = \underline{\underline{268}} \ (?)$$

Example

$$\lambda = (1^r) = \begin{array}{c} n \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} r \end{array}$$

$$\frac{\begin{array}{c} n \\ n-1 \\ \vdots \\ n-r+1 \end{array}}{\begin{array}{c} r \\ \vdots \\ 2 \\ 1 \end{array}} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \binom{n}{r}$$

$$\lambda = (\ell) = \overbrace{\boxed{\,\,\,}}^{\ell}$$

$$\frac{\boxed{n \quad n+1 \quad \cdots \quad n+\ell-1}}{\boxed{\ell \quad \cdots\cdots \quad 2 \quad 1}} = \binom{n+\ell-1}{\ell}$$

$$s_\lambda(x) = \frac{\det(x_i^{\ell_j})}{\Delta(x)} \qquad \ell_j = \lambda_j + n - j$$

$$t^{\lambda+\delta}$$

$$\delta_i = n - i$$

$$s_\lambda(t^\delta) = \frac{\det((t^{\delta_i})^{\ell_j})}{\Delta(t^\delta)} = \frac{\det((t^{\ell_j})^{\delta_i})}{\Delta(t^\delta)} = \frac{\Delta(t^\ell)}{\Delta(t^\delta)}$$

$$s_\lambda(t^\delta) = \frac{\Delta(t^{\lambda+\delta})}{\Delta(t^\delta)} = \prod_{1\le i<j\le n} \frac{t^{\lambda_i+\delta_i} - t^{\lambda_j+\delta_j}}{t^{\delta_i} - t^{\delta_j}} = \prod_{1\le i<j\le n} \frac{t^{\lambda_i+n-i} - t^{\lambda_j+n-j}}{t^{n-i} - t^{n-j}}$$

$$= \prod_{1\le i<j\le n} t^{\lambda_j} \frac{1 - t^{\lambda_i-\lambda_j+j-i}}{1 - t^{j-i}}$$

$$= t^{n(\lambda)} \prod_{1\le i<j\le n} \frac{1 - t^{\lambda_i-\lambda_j+j-i}}{1 - t^{j-i}}$$

$$\boxed{\prod_{1\le i<j\le n} \frac{1-t^{l_i-l_j}}{1-t^{\delta_i-\delta_j}}}$$

$$\| $$

$$\prod_{s\in\lambda} \frac{1-t^{n+c_\lambda(s)}}{1-t^{h_\lambda(s)}}$$

$$\prod_{1\le i<j\le n} (1-t^{l_i-l_j})$$

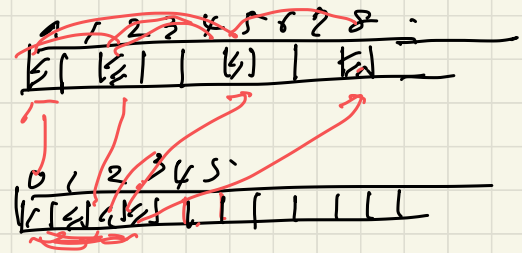
$$= \prod_{\substack{k<l \\ k,l\in M}} (1-t^{l-k})$$

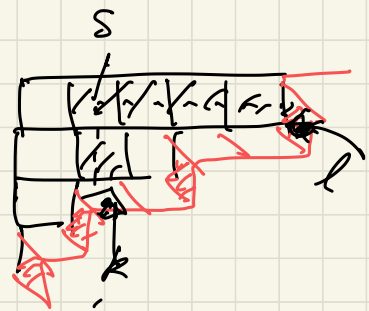
$l_i = \lambda_i + n - i$

$\lambda = (5,3,1,0)$

$l = (8,5,2,0)$

$l = \lambda + \delta$





hook

$k<l$

$k\notin M,\ l\in M$

$l-k = h_\lambda(s)$

⑥ Self-duality $(t=q)$

$$x = t^{\mu+\delta}$$

$$\frac{s_\lambda(t^{\mu+\delta})}{s_\lambda(t^{\delta})} = \frac{s_\mu(t^{\lambda+\delta})}{s_\mu(t^{\delta})} \qquad (\lambda, \mu \in \mathcal{P}_n)$$

$$s_\lambda(t^{\delta}) = \frac{\Delta(t^{\lambda+\delta})}{\Delta(t^{\delta})}$$

$$\det\left(\left(t^{\lambda_j+n-j}\right)^{\mu_i+n-i}\right)_{i,j=1}^{n}$$

$$\parallel$$

$$\frac{s_\lambda(t^{\mu+\delta})}{s_\lambda(t^{\delta})} = \frac{\Delta(t^{\delta})\det\left(\left(t^{\mu_i+n-i}\right)^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\Delta(t^{\lambda+\delta})\,\Delta(t^{\mu+\delta})} = \frac{s_\mu(t^{\lambda+\delta})}{s_\mu(t^{\delta})}$$

4° Cauchy's lemma

$x = (x_1, \dots, x_n) \quad y = (y_1, \dots, y_n)$

additive

$$(1) \quad \det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^{n} = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(x_i + y_j)} \qquad \frac{x_i \to x_i^{-1}}{}$$

multiplicative

$$(2) \quad \det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{n} = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1 - x_i y_j)}$$

Exercise: Prove!

Remark

(1) $$\det\left(\frac{x_i+y_j+\lambda}{(x_i+y_j)\lambda}\right)_{ij=1}^{n} = \frac{(\sum x_i+\sum y_j+\lambda)}{\lambda}\,\frac{\Delta(x)\Delta(y)}{\prod_{i,j}(x_i+y_j)}$$

rational

(2) $$\det\left(\frac{1-\lambda x_i y_j}{(1-x_i y_j)(1-\lambda)}\right)_{ij=1}^{n} = \frac{1-\lambda x_1\cdots x_n y_1\cdots y_n}{1-\lambda}\,\frac{\Delta(x)\Delta(y)}{\prod_{i,j}(1-x_i y_j)}$$

trigonometric

(3) $\sigma(z)$ = Weierstrass σ function

elliptic

$$\det\left(\frac{\sigma(x_i+y_j+\lambda)}{\sigma(x_i+y_j)\,\sigma(\lambda)}\right)_{ij=1}^{n} = \frac{\sigma(\sum x_i+\sum y_j+\lambda)}{\sigma(\lambda)}\,\frac{\prod_{1\le i<j\le n}\sigma(x_i-x_j)\sigma(y_i-y_j)}{\prod_{i,j}\sigma(x_i+y_j)}$$

Frobenius formula

Theorem (Cauchy formula)

$s_\lambda(x) = s_\lambda^{\det}(x)$

$$\prod_{i,j=1}^{n} \frac{1}{1-x_i y_j} = \sum_{\lambda \in \mathcal{P}_n} s_\lambda(x) s_\lambda(y)$$

$\mathbb{C}[[x,y]]$

Proof)

$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j}(1-x_i y_j)} = \det\left(\frac{1}{1-x_i y_j}\right)_{i,j=1}^{n}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \frac{1}{(1-x_{\sigma(1)}y_1)\cdots(1-x_{\sigma(n)}y_n)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{k_1, \dots, k_n \geq 0} (x_{\sigma(1)} y_1)^{k_1} \cdots (x_{\sigma(n)} y_n)^{k_n}$$

$$= \sum_{k_1 \dots k_n \geq 0} \underbrace{\left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) x_{\sigma(1)}^{k_1} \cdots x_{\sigma(n)}^{k_n} \right)} y_1^{k_1} \cdots y_n^{k_n} \quad \cdots (*)$$

$$\begin{cases} \Delta_{k_1 \cdots k_n}(x) = \det \left(x_i^{k_j} \right)_{i,j=1}^{n} \\ \Delta_{\underbrace{n-1, n-2, \dots, 0}_{\delta}}(x) = \Delta(x) \end{cases}$$

alternating w.r.t $(k_1, \dots, k_n)$

$$= \sum_{k_1 \dots k_n \geq 0} \Delta_{k_1 \cdots k_n}(x) \, y_1^{k_1} \cdots y_n^{k_n} \quad \rfloor$$

$k_1, \dots, k_n \geq 0$: mutually distinct

$$\Longrightarrow \exists! \; l_1 > \dots > l_n > 0 \quad \exists! \; \sigma \in \mathfrak{S}_n \quad \Delta_{l_1 \cdots l_n}(y)$$

$$(k_1, \dots, k_n) = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$$

$$\Delta_{k_1 \cdots k_n}(x) = \Delta_{l_{\sigma(1)} \cdots l_{\sigma(n)}}(x) = \operatorname{sgn}(\sigma) \Delta_{l_1 \cdots l_n}(x)$$

$$\therefore (*) = \sum_{l_1 > \dots > l_n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \Delta_{l_1 \cdots l_n}(x) \, y_1^{l_{\sigma(1)}} \cdots y_n^{l_{\sigma(n)}}$$

$$= \sum_{l_1 > \dots > l_n \geq 0} \Delta_{l_1 \cdots l_n}(x) \boxed{\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \, y_1^{l_{\sigma(1)}} \cdots y_n^{l_{\sigma(n)}}}$$

$$\frac{\Delta(x)\Delta(y)}{\prod_{ij}(1-x_i y_j)} = \det\left(\frac{1}{1-x_i y_j}\right)$$

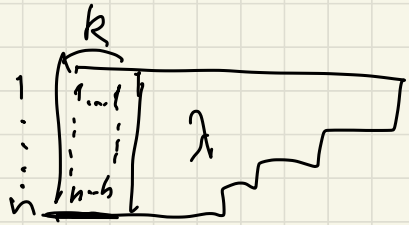
$$= \sum_{k_1 \cdots k_n \geq 0} \Delta_{k_1 \cdots k_n}(x)\, y_1^{k_1} \cdots y_n^{k_n}$$

$$= \sum_{l_1 > \cdots > l_n \geq 0} \Delta_{l_1 \cdots l_n}(x)\Delta_{l_1 \cdots l_n}(y)$$

$$\frac{1}{\prod_{ij}(1-x_i y_j)} = \sum_{l_1 > \cdots > l_n \geq 0} \underbrace{\frac{\Delta_{l_1 \cdots l_n}(x)}{\Delta(x)}}_{S_\lambda(x)} \underbrace{\frac{\Delta_{l_1 \cdots l_n}(y)}{\Delta(y)}}_{S_\lambda(y)}$$

5° Equivalence of the two definitions.

$$s_\lambda^{comb}(x) = s_\lambda^{det}(x).$$

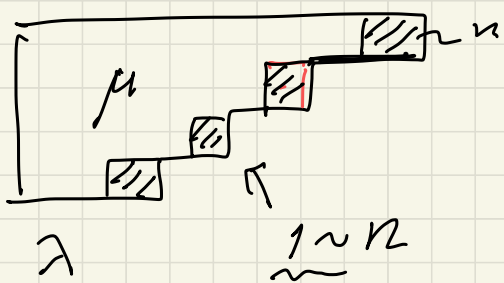
- 

$(x_1 \cdots x_n)^k$

$$s^{comb}_{\lambda+(k^n)}(x) = \underline{(x_1 \cdots x_n)}^k \, s^{comb}_\lambda(x)$$

$$\lambda + (k^n) = (k, k \cdots k)$$

$$s^{det}_{\lambda+(k^n)}(x) = (x_1 \cdots x_n)^k \, s^{det}_\lambda(x)$$

- $T =$ 

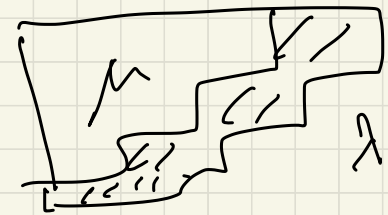
T' = SSTableau consisting of $1 \sim n-1$ shape μ

μ : partition

horizontal strip

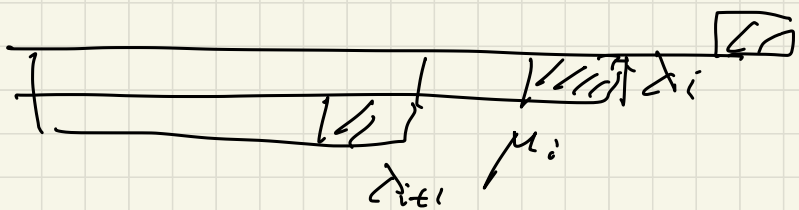
$\lambda = (\lambda_1, \lambda_2, \dots)$ $\quad \mu = (\mu_1, \mu_2, \dots)$

$\mu \leq \lambda \quad (\mu_i \leq \lambda_i)$



λ/μ : skew diagram

λ/μ : horizontal strip $\iff$ $\underline{\lambda_i \geq \mu_i \geq \lambda_{i+1} \quad (i \geq 1)}$



$$s_\lambda^{comb}(x) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu:\ \text{horizontal strip}}} s_\mu^{comb}(x_1, \dots, x_{n-1})\, x_n^{|\lambda|-|\mu|}$$

recurren formula

$$s_\lambda^{det}(x) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu:\ h.s}} s_\mu^{det}(x_1, \dots, x_{n-1})\, x_n^{|\lambda|-|\mu|}$$

$$s_\lambda^{comb}(x) = s_\lambda^{det}(x)$$

λ

$(x_1 \dots x_n)^{\lambda}$

We have only to prove this for λ with $\lambda_n = 0$.

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$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j}(1-x_i y_j)} = \sum_{l_1 \cdots l_n \geq 0} \Delta_{l_1 \cdots l_n}(x)\, y_1^{l_1} \cdots y_n^{l_n}$$

$$\frac{\Delta(y)}{\prod_{i,j}(1-x_i y_j)} = \sum_{l_1 \cdots l_n \geq 0} \frac{\Delta_{l_1 \cdots l_n}(x)}{\Delta(x)}\, y_1^{l_1} \cdots y_n^{l_n}$$

$$\| \qquad l_1 > \cdots > l_n \geq 0$$

$$f_{l_1 \cdots l_n}(x) \qquad s_\lambda(x)$$

$$l_1 > \cdots > l_n \geq 0 \Rightarrow \underline{f_{l_1 \cdots l_n}(x) = s_\lambda(x)}$$

$$\sum_{l_1 \cdots l_n \geq 0} f_{l_1 \cdots l_n}(x)\, y_1^{l_1} \cdots y_n^{l_n} = \frac{\Delta(y)}{\prod_{i,j=1}^{n} (1 - x_i y_j)} \qquad (*)$$

$\boxed{y_n = 0}$

$$\sum_{l_1 \cdots l_{n-1} \geq 0} f_{l_1 \cdots l_{n-1} 0}(x)\, y_1^{l_1} \cdots y_{n-1}^{l_{n-1}} = \boxed{\frac{\Delta(y_1 \cdots y_{n-1})}{\prod_{i,j=1}^{n-1} (1 - x_i y_j)}} \frac{y_1 \cdots y_{n-1}}{\prod_{j=1}^{n-1} (1 - x_n y_j)}$$

$$= \left(\sum_{k_1 \cdots k_{n-1}} f_{k_1 \cdots k_{n-1}}(x_1 \cdots x_{n-1})\, y_1^{k_1} \cdots y_{n-1}^{k_{n-1}} \right) \left(\sum_{r_1 \cdots r_{n-1} \geq 0} x_n^{\sum r_j}\, y_1^{r_1+1} \cdots y_n^{r_n+1} \right).$$

$$\underline{f_{l_1 \dots l_{n-1} 0}}(x) = \sum_{\substack{(k_1 \dots k_{n-1}) \\ \begin{cases} 0 \le k_1 < l_1 \\ \vdots \\ 0 \le k_{n-1} < l_{n-1} \end{cases}}} \frac{\underline{f_{k_1 \dots k_{n-1}}}(x_1 \dots x_{n-1})\, x_n^{\sum_i l_i - \sum_i k_i - (n-1)}}{}$$

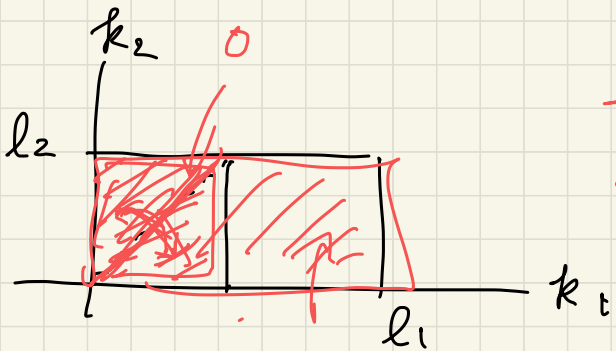
alternating

$$[a, b) = \{ k \in \mathbb{Z} \mid a \le k < b \}$$

$$(k_1 \dots k_{n-1}) \in \underline{[0, l_1) \times [0, l_2) \times \dots \times [0, l_{n-1})}$$

$$l_1 > l_2 > \dots > l_{n-1} \ge 0$$

(k_1, k_2)



$$\sum_{[0, l_1) \times [0, l_2)} \quad (*)$$

$$= \sum_{[k_1, l_1) \times [0, l_2)} \quad (**)$$

$$f_{l_1 \cdots l_{n-1} 0}(x_1 \cdots x_n) = \sum_{(k_1 \cdots k_{n-1}) \in [0,l_1) \times [0,l_2) \times \cdots \times [0,l_{n-1}]} f_{k_1 \cdots k_{n-1}}(x_1 \cdots x_{n-1})\, x_n$$

$l_1 > \cdots > l_{n-1} \geq 0$

$\sum l_i - \sum k_i - (n-1)$

$|\lambda| - |\mu|$

$S_\lambda(x_1 \cdots x_n)$

$S_\mu(x_1 \cdots x_{n-1})$

$$= \sum_{(k_1 \cdots k_n) \in [l_2, l_1) \times [l_3, l_2) \times \cdots \times [0, l_{n-1})}$$

$\lambda \in \mathcal{P}_n$. $l_1 > k_1 \geq l_2 > k_2 \geq \cdots \geq l_{n-1} > k_{n-1} \geq 0$

$l_i = \lambda_i + n - i$

$\mu \in \mathcal{P}_{n-1}$ $k_i = \mu_i + n - i - 1$

$\Downarrow$

$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq 0$

horizontal strip