Introduction to Macdonald polynomials: Lecture 4

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2 Macdonald polynomials: Definition and examples

2.1 Macdonald-Ruijsenaars q-difference operator

We fix $q, t \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ with $|q| < 1$. We regard the variables $x = (x_1, \ldots, x_n)$ as the canonical coordinates of the n-dimensional algebraic torus $(\mathbb{C}^*)^n$.

The *Macdonald-Ruijsenaars q-difference operator* of first order with parameter t is defined by

$$D_x = \sum_{i=1}^{n} A_i(x) T_{q,x_i} = \sum_{i=1}^{n} \prod_{1\le j\le n;\, j\ne i} \frac{tx_i - x_j}{x_i - x_j}\, T_{q,x_i}, \tag{2.1}$$

where T_{q,x_i} stands for the q-shift operator in the variable x_i:

$$T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{2.2}$$

We remark that the coefficients of D_x are expressed as

$$A_i(x) = \prod_{j\ne i} \frac{tx_i - x_j}{x_i - x_j} = \frac{T_{t,x_i}\Delta(x)}{\Delta(x)}, \qquad \Delta(x) = \prod_{1\le i<j\le n} (x_i - x_j) \tag{2.3}$$

in terms of the difference product $\Delta(x)$ of x.

• **Fundamental properties of D_x**

(1) D_x is invariant under the action of $\mathfrak{S}_n$ such that $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(T_{q,x_i}) = T_{q,x_{\sigma(i)}}$ $(\sigma \in \mathfrak{S}_n)$.

(2) $D_x :\ \mathbb{C}(x) \to \mathbb{C}(x)$ stabilizes $\mathbb{C}[x]_n^{\mathfrak{S}}$, i.e. $D_x(\mathbb{C}[x]^{\mathfrak{S}_n}) \subseteq \mathbb{C}[x]^{\mathfrak{S}_n}$. Warning: $D_x(\mathbb{C}[x]) \not\subseteq \mathbb{C}[x]$.

(3) $D_x :\ \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is triangular with respect to the dominance order of $m_\lambda(x)$:

$$D_x m_\lambda(x) = \sum_{\mu\le\lambda} d_\mu^\lambda\, m_\mu(x) = d_\lambda\, m_\lambda(x) + \sum_{\mu<\lambda} d_\mu^\lambda\, m_\mu(x) \qquad (\lambda \in \mathcal{P}_n). \tag{2.4}$$

where $d_\lambda = d_\lambda^\lambda = \sum_{i=1}^n t^{n-i} q^{\lambda_i}$.

Theorem (Macdonald) Suppose that the parameters $q, t \in \mathbb{C}^*$ are generic. Then, for each partition $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) = P_\lambda(x|q,t) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that

$$(1)\ \ D_x P_\lambda(x) = d_\lambda P_\lambda(x) \qquad (2)\ \ P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} u_\mu^\lambda\, m_\mu(x) \quad (u_\mu^\lambda \in \mathbb{C}). \tag{2.5}$$

The eigenfunction $P_\lambda(x) \in \mathbb{C}[x]_n^{\mathfrak{S}}$ is called the *Macdonald polynomial* attached to the partition $\lambda \in \mathcal{P}_n$. We remark that the genericity condition is fulfilled if $1, t, \ldots, t^{n-1}$ are linearly independent over $\mathbb{Q}(q)$, for example. If we regard q, t as indeterminates, $P_\lambda(x|q,t)$ is determined as a unique symmetric polynomial in $\mathbb{Q}(q,t)[x]^{\mathfrak{S}_n}$.

• **Comments on the proof:** The coefficients u^λ_μ of $P_\lambda(x)$ are determined through recurrence formulas

$$(d_\lambda - d_\mu) u^\lambda_\mu = \sum_{\mu<\nu\le\lambda} u^\lambda_\nu \, d^\nu_\mu \qquad (\mu < \lambda) \tag{2.6}$$

by the descending induction with respect to $\le$, provided that $d_\mu \ne d_\lambda$ $(\mu < \lambda)$. The eigenfunction $P_\lambda(x)$ can also be expressed as

$$P_\lambda(x) = \prod_{\mu<\lambda} \frac{D_x - d_\mu}{d_\lambda - d_\mu} (m_\lambda(x)). \tag{2.7}$$

• **Single columns:** If λ is a single column $(1^r) = (1, \ldots, 1, 0, \ldots, 0)$ $(r = 0, 1, \ldots, n)$, we have $P_{(1^r)}(x) = e_r(x)$ (Why?). The equation $D_x e_r(x) = d_{(1^r)} e_r(x)$ already implies a nontrivial identity

$$\begin{aligned} &\sum_{i=1}^n \prod_{j\ne i} \frac{t x_i - x_j}{x_i - x_j} e_r(x_1, \ldots, q x_i, \ldots, x_n) = d_{(1^r)} e_r(x), \\ &d_{(1^r)} = t^{n-1} q + \cdots + t^{n-r} q + t^{n-r-1} + \cdots + 1 = q t^{n-r} \frac{1 - t^r}{1 - t} + \frac{1 - t^{n-r}}{1 - t}. \end{aligned} \tag{2.8}$$

In particular, $D_x(1) = d_0 1$ implies

$$\sum_{i=1}^n \prod_{j\ne i} \frac{t x_i - x_j}{x_i - x_j} = \frac{1 - t^n}{1 - t}. \tag{2.9}$$

• **Adding columns of length n:** Similarly to the case of Schur functions, we have

$$P_{\lambda+(k^n)}(x) = (x_1 \cdots x_n)^k P_\lambda(x) \qquad (\lambda \in \mathcal{P}_n;\ k = 0, 1, 2, \ldots). \tag{2.10}$$

2.2 Some examples

• $t = 1$: $P_\lambda(x) = m_\lambda(x)$ (monomial symmetric function). $D_x = \sum_{i=1}^n T_{q,x_i}$

• $t = q$: $P_\lambda(x) = s_\lambda(x)$ (Schur function). When $t = q$, we have

$$D_x = \sum_{i=1}^n \frac{T_{q,x_i}(\Delta(x))}{\Delta(x)} T_{q,x_i} = \frac{1}{\Delta(x)} \Big(\sum_{i=1}^n T_{q,x_i} \Big) \Delta(x). \tag{2.11}$$

• $n = 1$: $P_{(l)}(x_1) = x_1^l$ $(l = 0, 1, 2, \ldots)$.

• $n = 2$: For each $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_2$, $P_\lambda(x)$ is determined explicitly as follows:

$$\begin{aligned} &P_{(\lambda_1,\lambda_2)}(x_1, x_2) = (x_1 x_2)^{\lambda_2} P_{(l,0)}(x_1, x_2); \\ &P_{(l.0)}(x_1, x_2) = \frac{(q;q)_l}{(t;q)_l} \sum_{\mu_1+\mu_2=l} \frac{(t;q)_{\mu_1} (t;q)_{\mu_2}}{(q;q)_{\mu_1} (q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2} = \frac{(q;q)_l}{(t;q)_l} Q_{(l,0)}(x_1, x_2). \end{aligned} \tag{2.12}$$

where $(t;q)_k = (1-t)(1-qt) \cdots (1-q^{k-1}t)$ $(k = 0, 1, 2, \ldots)$

2.3 Eigenfunctions in the case where $n = 2$

For $\mu = (\mu_1, \mu_2), \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^n$, we have

$$\begin{aligned} \mu \leq \lambda \quad &\iff \quad (\mu_1 \leq \lambda_1,\ \mu_1 + \mu_2 = \lambda_1 + \lambda_2) \\ &\iff \quad (\mu_1, \mu_2) = (\lambda_1 - k, \lambda_2 + k) \quad (k \in \mathbb{Z}_{\geq 0}). \end{aligned} \tag{2.13}$$

In view of this, we consider a formal power series of the form

$$\varphi(x_1, x_2) = \sum_{k \geq 0} c_k x_1^{\lambda_1 - k} x_2^{\lambda_2 + k} = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{k \geq 0} c_k (x_2/x_1)^k \tag{2.14}$$

with $c_0 = 1$, and solve the eigenfunction equation

$$\frac{tx_1 - x_2}{x_1 - x_2} \varphi(qx_1, x_2) + \frac{x_1 - tx_2}{x_1 - x_2} \varphi(x_1, qx_2) = \varepsilon\, \varphi(x_1, x_2). \tag{2.15}$$

Setting $z = x_2/x_1$, we rewrite this equation by means of $f(z) = \sum_{k \geq 0} c_k z^k$. Since $\varphi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} f(x_2/x_1)$, we obtain

$$\frac{t - z}{1 - z} q^{\lambda_1} f(q^{-1}z) + \frac{1 - tz}{1 - z} q^{\lambda_2} f(qz) = \varepsilon f(z). \tag{2.16}$$

namely,

$$(t - z) q^{\lambda_1} f(q^{-1}z) + (1 - tz) q^{\lambda_2} f(qz) = \varepsilon (1 - z) f(z), \tag{2.17}$$

This equation gives rise to the recurrence formulas for the coefficients

$$(tq^{\lambda_1 - k} + q^{\lambda_2 + k} - \varepsilon) c_k = (q^{\lambda_1 - k + 1} + tq^{\lambda_2 + k - 1} - \varepsilon) c_{k-1} \quad (k \in \mathbb{Z}), \tag{2.18}$$

with $c_k = 0$ for $k < 0$. This formula for $k = 0$ determines the eigenvalue as $\varepsilon = tq^{\lambda_1} + q^{\lambda_2}$. Then the resulting recurrence formulas

$$(1 - q^k)(1 - q^{\lambda_2 - \lambda_1 + k}/t) c_k = (q/t)(1 - tq^{k-1})(1 - q^{\lambda_2 - \lambda_1 + k - 1}) c_{k-1} \quad (k = 1, 2, \ldots) \tag{2.19}$$

are solved by

$$c_k = \frac{(t; q)_k (q^{\lambda_2 - \lambda_1}; q)_k}{(q; q)_k (q^{\lambda_2 - \lambda_1 + 1}/t; q)_k} (q/t)^k \quad (k = 0, 1, 2, \ldots) \tag{2.20}$$

This computation implies that the power series

$$\begin{aligned} \varphi(x_1, x_2) &= x_1^{\lambda_1} x_2^{\lambda_2} \sum_{k=0}^{\infty} \frac{(t; q)_k (q^{\lambda_2 - \lambda_1}; q)_k}{(q; q)_k (q^{\lambda_2 - \lambda_1 + 1}/t; q)_k} (qx_2/tx_1)^k \\ &= x_1^{\lambda_1} x_2^{\lambda_2}\, {}_2\phi_1\!\left[\begin{matrix} t,\ q^{\lambda_2 - \lambda_1} \\ q^{\lambda_2 - \lambda_1 + 1}/t \end{matrix}; q,\ qx_2/tx_1\right] \end{aligned} \tag{2.21}$$

solves the eigenfunction equation $D_x \varphi(x) = \varepsilon\, \varphi(x)$ with $\varepsilon = tq^{\lambda_1} + q^{\lambda_2}$.

- ***q*-Hypergeometric series**

$${}_{r+1}\phi_r\!\left[\begin{matrix} a_0,\ a_1,\ \ldots,\ a_r \\ b_1,\ \ldots,\ b_r \end{matrix}; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_0; q)_k (a_1; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_r; q)_k} z^k \quad (|z| < 1). \tag{2.22}$$

- **q-Binomial theorem**

$$
{}_1\phi_0\left[\begin{matrix} a \\ \cdot \end{matrix} ; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|z|<1), \quad (z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z) \quad (|q|<1). \tag{2.23}
$$

- **Generating function**

$$
\begin{aligned}
&\frac{(tx_1y : q)_\infty (tx_2y : q)_\infty}{(x_1y;q)_\infty (x_2y;q)_\infty} = \sum_{l=0}^{\infty} Q_{(l)}(x_1,x_2) y^l. \\
&Q_{(l)}(x_1,x_2) = \frac{(t;q)_l}{(q;q)_l} P_{(l)}(x_1,x_2) = \sum_{\mu_1+\mu_2=l} \frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2}
\end{aligned} \tag{2.24}
$$

2.4 Macdonald polynomials attached to single rows

For $l = 0, 1, 2, \ldots$, we set

$$
Q_{(l)}(x) = \sum_{\mu_1+\cdots+\mu_n=l} \frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n} = \frac{(t;q)_l}{(q;q)_l} x_1^l + (\text{lower order terms}). \tag{2.25}
$$

These polynomials are the Macdonald polynomials attached to single rows up to constant multiples. If fact we have

$$
Q_{(l)}(x) = \frac{(t;q)_l}{(q;q)_l} P_{(l)}(x) \quad (l = 0, 1, \ldots). \tag{2.26}
$$

In order to prove this, we have only to show that

$$
D_x Q_{(l)}(x) = d_{(l)} Q_{(l)}(x), \quad d_{(l)} = t^{n-1}q^l + t^{n-2} + \cdots + 1 = t^{n-1}q^l + \frac{1-t^{n-1}}{1-t}. \tag{2.27}
$$

By the q-binomial theorem, we have

$$
\begin{aligned}
\Phi(x;y) &= \frac{(tx_1y;q)_\infty \cdots (tx_ny;q)_\infty}{(x_1y;q)_\infty \cdots (x_ny;q)_\infty} \\
&= \sum_{\mu_1,\ldots,\mu_n \geq 0} \frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} \cdots \frac{(t;q)_{\mu_n}}{(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n} y^{\mu_1+\cdots+\mu_n} = \sum_{l=0}^{\infty} Q_{(l)}(x) y^l.
\end{aligned} \tag{2.28}
$$

Since

$$
\Phi(x;y) = \sum_{l=0}^{\infty} Q_{(l)}(x) y^l, \tag{2.29}
$$

the eigenfunction equations for $Q_{(l)}(x)$ are equivalent to the identify

$$
D_x \Phi(x;y) = \Big(t^{n-1} T_{q,y} + \frac{1-t^{n-1}}{1-t}\Big)\Phi(x;y). \tag{2.30}
$$

Noting that

$$
T_{q,x_i}\Phi(x;y) = \frac{1-x_iy}{1-tx_iy}\Phi(x;y), \qquad T_{q,y}\Phi(x;y) = \prod_{j=1}^{n} \frac{1-x_jy}{1-tx_jy}\Phi(x;y), \tag{2.31}
$$

we see that identity (2.30) is equivalent to the identity

$$\sum_{i=1}^{n}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\frac{1-x_iy}{1-tx_iy}=t^{n-1}\prod_{j=1}^{n}\frac{1-x_jy}{1-tx_jy}+\frac{1-t^{n-1}}{1-t} \tag{2.32}$$

of rational functions. As rational functions of y, both sides are of the form

$$\frac{p(y)}{q(y)},\quad p(y),q(y)\in\mathbb{C}[y],\quad \deg_y p(y)\leq n,\quad \deg_y q(y)=n. \tag{2.33}$$

Then, one can verify identity (2.32) by partial fraction expansions, comparing the residues at $y=1/t_ix_i$ $(i=1,\ldots,n)$ and the value at $y=0$ which reduces to (2.9).

• **Relation to q-hypergeoemtric series**

$$Q_{(l)}(x)=\sum_{\mu_1+\cdots+\mu_n=l}\frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}}x_1^{\mu_1}\cdots x_n^{\mu_n} \tag{2.34}$$

Using $\mu_1=l-\mu_2-\cdots-\mu_n$, we rewrite the factor containing μ_1:

$$\begin{aligned}\frac{(t:q)_{\mu_1}}{(q;q)_{\mu_1}}&=\frac{(t;q)_{l-\mu_2-\cdots-\mu_n}}{(q;q)_{l-\mu_2-\cdots-\mu_n}}=\frac{(t;q)_l}{(q;q)_l}\frac{(q^{l-\mu_2-\cdots-\mu_n+1};q)_{\mu_2+\cdots+\mu_n}}{(q^{l-\mu_2-\cdots-\mu_n}t;q)_{\mu_2+\cdots+\mu_n}}\\&=\frac{(t;q)_l}{(q;q)_l}\frac{(q^{-l};q)_{\mu_2+\cdots+\mu_n}}{(q^{-l+1}/t;q)_{\mu_2+\cdots+\mu_n}}(q/t)^{\mu_2+\cdots+\mu_n}\end{aligned} \tag{2.35}$$

Hence we have

$$Q_{(l)}(x)=\frac{(t;q)_l}{(q;q)_l}x_1^l\sum_{\mu_2,\ldots,\mu_n\geq 0}\frac{(q^{-l};q)_{\mu_2+\cdots+\mu_n}}{(q^{-l+1}/t;q)_{\mu_2+\cdots+\mu_n}}\frac{(t;q)_{\mu_2}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_2}\cdots(q;q)_{\mu_n}}(qx_2/tx_1)^{\mu_2}\cdots(qx_n/tx_1)^{\mu_n}. \tag{2.36}$$

We now introduce a q-hypergeometric series in m variables

$$\phi_D\left[\begin{matrix}a;\ b_1,\ldots,b_m\\ c\end{matrix};\ q;\ z_1,\ldots,z_m\right]=\sum_{k_1,\ldots,k_m\geq 0}\frac{(a;q)_{k_1+\cdots+k_m}}{(c;q)_{k_1+\cdots+k_m}}\frac{(b_1;q)_{k_1}\cdots(b_m;q)_{k_m}}{(q;q)_{k_1}\cdots(q;q)_{k_m}}z_1^{k_1}\cdots z_m^{k_m}, \tag{2.37}$$

which is a q-analogue of Lauricella's F_D (Appell's F_1 when $m=2$). Then the Macdonald polynomials attached to single rows are expressed as

$$P_{(l)}(x_1,\ldots,x_n)=x_1^l\phi_D\left[\begin{matrix}q^{-l};\ t,\ldots,t\\ q^{-l+1}/t\end{matrix};\ q;\ qx_2/tx_1,\ldots,qx_n/tx_1\right]\quad(l=0,1,2,\ldots) \tag{2.38}$$

in terms of ϕ_D in $n-1$ variables.