# Introduction to Macdonald polynomials: Lecture 5 

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## 2 Macdonald polynomials: Definition and examples

### 2.3 Eigenfunctions in the case where $n=2$ (continued)

### 2.4 Macdonald polynomials attached to single rows

(See the summary of Lecture 4.)

## 3 Orthogonality relations

### 3.1 Scalar product and orthogonality

We define a meromorphic function $w(x)$ on $\left(\mathbb{C}^{*}\right)^{n}$ by

$$
\begin{equation*}
w(x)=\prod_{1 \leq i<j \leq n} \frac{\left(x_{i} / x_{j} ; q\right)_{\infty}}{\left(t x_{i} / x_{j} ; q\right)_{\infty}} \frac{\left(x_{j} / x_{i} ; q\right)_{\infty}}{\left(t x_{j} / x_{i} ; q\right)_{\infty}}, \quad(z ; q)_{\infty}=\prod_{i=0}^{\infty}\left(1-q^{i} z\right) \quad(z \in \mathbb{C},|q|<1) . \tag{3.1}
\end{equation*}
$$

We assume $|t|<1$ so that $w(x)$ is holomorphic in a neighborhood of the $n$-dimensional torus

$$
\begin{equation*}
\mathbb{T}^{n}=\left\{x=\left(x_{1}, \ldots, x_{n}\right) \in\left(\mathbb{C}^{*}\right)^{n}| | x_{i} \mid=1(i=1, \ldots, n)\right\} . \tag{3.2}
\end{equation*}
$$

For a pair of holomprhic functions $f(x), g(x)$ in a neighborhood of $\mathbb{T}^{n}$, we define the scalar product (symmetric bilinear form) $\langle f, g\rangle$ by

$$
\begin{equation*}
\langle f, g\rangle=\frac{1}{n!} \frac{1}{(2 \pi \sqrt{-1})^{n}} \int_{\mathbb{T}^{n}} f\left(x^{-1}\right) g(x) w(x) \frac{d x_{1} \cdots d x_{n}}{x_{1} \cdots x_{n}}=\frac{1}{n!} \mathrm{CT}\left[f\left(x^{-1}\right) g(x) w(x)\right], \tag{3.3}
\end{equation*}
$$

where CT denotes the constant term of the Laurent expansion of a holomorphic function around $\mathbb{T}^{n}$. Then the Macdonald polynomials are orthogonal with respect to this scalar product:

$$
\begin{equation*}
\left\langle P_{\lambda}(x), P_{\mu}(x)\right\rangle=\delta_{\lambda, \mu} N_{\lambda} \quad\left(\lambda, \mu \in \mathcal{P}_{n}\right) . \tag{3.4}
\end{equation*}
$$

- The constant term and the scalar products are determined explicitly as follow:

$$
\begin{align*}
& N_{\phi}=\left(\frac{(t ; q)_{\infty}}{(q ; q)_{\infty}}\right)^{n} \prod_{i=1}^{n} \frac{\left(t^{i-1} q ; q\right)_{\infty}}{\left(t^{i} ; q\right)_{\infty}}=\prod_{1 \leq i<j \leq n} \frac{\left(t^{j-i} ; q\right)_{\infty}\left(q t^{j-i} ; q\right)_{\infty}}{\left(t^{j-i+1} ; q\right)_{\infty}\left(q t^{j-i-1} ; q\right)_{\infty}}, \\
& N_{\lambda}=\prod_{1 \leq i<j \leq n} \frac{\left(q^{\lambda_{i}-\lambda_{j}} t^{j-i} ; q\right)_{\infty}\left(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i} ; q\right)_{\infty}}{\left(q^{\lambda_{i}-\lambda_{j}} t^{j-i+1} ; q\right)_{\infty}\left(q^{\lambda_{i}-\lambda_{j}+1} t^{j-i-1} ; q\right)_{\infty}} \tag{3.5}
\end{align*}
$$

Research project: How can one derive these explicit formulas? Variations and generalizations?

- If $q, t \in \mathbb{R}$ and $|q|<1,|t|<1$, the Macdonald polynomials have real coefficients, and $\langle$,$\rangle defines$ a positive definite scalar product on $\mathbb{R}[x]^{\mathfrak{G}_{n}}$.


### 3.2 Comments on the orthogonality relation

The orthogonality of the Macdonald polynomials is a consequence of the facts that
(1) The $q$-difference operator $D_{x}$ is (formally) self-adjoint w.r.t. the weight function $w(x)$,
(2) The Macdonald polynomials are separated by the eigenvalues of $D_{x}$ (for generic $t$ ).

- Cauchy's theorem as a basis of $q$-difference de Rham theory: Let $\varphi(z)$ be a holomorphic function in an neighborhood of a closed curve $C$ in $\mathbb{C}^{*}$. Then we have

$$
\begin{equation*}
\int_{C} T_{q, z}(\varphi(z)) \frac{d z}{z}=\int_{C} \varphi(z) \frac{d z}{z}, \quad \text { i.e. } \int_{C}\left(T_{q, z}-1\right)(\varphi(z)) \frac{d z}{z}=0 \tag{3.6}
\end{equation*}
$$

if $q$ is sufficiently close to 1 (so that $C$ can be deformed continuously to $q C$ in a domain where $\varphi(z)$ is holomorphic). In particular we have

$$
\begin{equation*}
\int_{C} T_{q, z}(\varphi(z)) \psi(z) \frac{d z}{z}=\int_{C} \varphi(z) T_{q, z}^{-1}(\psi(z)) \frac{d z}{z} \tag{3.7}
\end{equation*}
$$

This formula play the role of the integration by parts.

- Formal adjoint of a $q$-difference operator: Let $L \in \mathbb{C}(x)\left[T_{q, x}^{ \pm 1}\right]$ be a $q$-difference operator in $x=\left(x_{1}, \ldots, x_{n}\right)$ with rational coefficients of the form $L_{x}=\sum_{\mu \in \mathbb{Z}^{n}} a_{\mu}(x) T_{q, x}^{\mu}$ (finite sum), where $T_{q, x}^{\mu}=T_{q, x_{1}}^{\mu_{1}} \cdots T_{q, x_{n}}^{\mu_{n}}$. We define the formal adjoint $L^{*}$ of $L$ by $L_{x}^{*}=\sum_{\mu \in \mathbb{Z}^{\mu}} T_{q, x}^{-\mu} a_{\mu}(x)$, so that $\left(L_{x} M_{x}\right)^{*}=M_{x}^{*} L_{x}^{*}$. Then, we have

$$
\begin{align*}
\int_{\mathbb{T}^{n}}\left(L_{x} f\right)\left(x^{-1}\right) g(x) w(x) \frac{d x}{x} & =\int_{\mathbb{T}^{n}} L_{x^{-1}}\left(f\left(x^{-1}\right)\right) g(x) w(x) \frac{d x}{x} \\
& =\int_{\mathbb{T}^{n}} f\left(x^{-1}\right) L_{x^{-1}}^{*}(g(x) w(x)) \frac{d x}{x}  \tag{3.8}\\
& =\int_{\mathbb{T}^{n}} f\left(x^{-1}\right) w(x)^{-1} L_{x^{-1}}^{*}(w(x) g(x)) w(x) \frac{d x}{x}
\end{align*}
$$

and

$$
\begin{equation*}
\langle L f, g\rangle=\left\langle f, L^{\dagger} g\right\rangle, \quad L^{\dagger}=w(x)^{-1} L_{x^{-1}}^{*} w(x), \tag{3.9}
\end{equation*}
$$

provided that $q$ is sufficiently close to 1 and that Cauchy's theorem can be applied to $L_{x}$. We say that $L_{x}$ is formally self-adjoint with respect to $w(x)$ if $L_{x}^{\dagger}=L_{x}$, namely $w(x) L_{x} w(x)^{-1}=L_{x^{-1}}^{*}$.

- $D_{x}$ is formally self-adjoint with respect to $\boldsymbol{w}(x)$ : Note that

$$
\begin{equation*}
\frac{T_{q, x_{i}} w(x)}{w(x)}=\prod_{j \neq i} \frac{1-t x_{i} / x_{j}}{1-x_{i} / x_{j}} \prod_{j \neq i} \frac{1-x_{j} / q x_{i}}{1-t x_{j} / q x_{i}}=\frac{A_{i}(x)}{T_{q, x_{i}} A_{i}\left(x^{-1}\right)} \quad(i=1, \ldots, n) . \tag{3.10}
\end{equation*}
$$

This implies that

$$
\begin{align*}
w(x) D_{x} w(x)^{-1} & =\sum_{i=1}^{n} A_{i}(x) \frac{w(x)}{T_{q, x_{i}} w(x)} T_{q, x_{i}}=\sum_{i=1}^{n} T_{q, x_{i}}\left(A_{i}\left(x^{-1}\right)\right) T_{q, x_{i}}  \tag{3.11}\\
& =\sum_{i=1}^{n} T_{q, x_{i}} A_{i}\left(x^{-1}\right)=D_{x^{-1}}^{*} .
\end{align*}
$$

It can be verified directly that $\left\langle D_{x} f, g\right\rangle=\left\langle f, D_{x} g\right\rangle$ if $|t|<|q|<1$. (Note that the poles of $A_{i}(x)$ along $\Delta(x)=0$ are canceled by the zeros of $w(x)$.)

- Orthogonality: For $\lambda, \mu \in \mathcal{P}_{n}$, the equality $\left\langle D_{x} P_{\lambda}(x), P_{\mu}(x)\right\rangle=\left\langle P_{\lambda}(x), D_{x} P_{\mu}(x)\right\rangle$ implies $d_{\lambda}\left\langle P_{\lambda}, P_{\mu}\right\rangle=d_{\mu}\left\langle P_{\lambda}, P_{\mu}\right\rangle$. If $t$ is generic, we have $d_{\lambda} \neq d_{\mu}(\lambda \neq \mu)$, and hence $\left\langle P_{\lambda}, P_{\mu}\right\rangle=0(\lambda \neq \mu)$.


## 4 Commuting family of $q$-difference operators

### 4.1 Macdonald-Ruijsenaars operators of higher order

For each $r=0,1, \ldots, n$, we define the Macdonald-Ruijsenaars operator of $r$ th order by

$$
\begin{equation*}
D_{x}^{(r)}=\sum_{I \subseteq\{1, \ldots, n\} ;|I|=r} A_{I}(x) T_{q, x}^{I}=\sum_{I \subseteq\{1, \ldots, n\} ;|I|=r} t^{\binom{r}{2}} \prod_{i \in I, j \notin I} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}} \prod_{i \in I} T_{q, x_{i}}, \tag{4.1}
\end{equation*}
$$

where $T_{q, x}^{I}=\prod_{i \in I} T_{q, x_{i}}$, so that $D_{x}^{(0)}=1, D_{x}^{(1)}=D_{x}$ and $D_{n}^{(n)}=t^{\binom{n}{2}} T_{q, x_{1}} \cdots T_{q, x_{n}}$. Note also that

$$
\begin{equation*}
A_{I}(x)=t^{\binom{|I|}{2}} \prod_{i \in I, j \neq I} \frac{t x_{i}-x_{j}}{x_{i}-x_{j}}=\frac{T_{t, x}^{I} \Delta(x)}{\Delta(x)} . \tag{4.2}
\end{equation*}
$$

It is known by Ruijsenaars (1987) that the $q$-difference operators $D_{x}^{(r)}(r=1, \ldots, n)$ commute with each other. By the same method we applied to $D_{x}$, one can directly verify:
(1) $D_{x}^{(r)}(r=1, \ldots, n)$ are invariant under the action of $\mathfrak{S}_{n}$.
(2) $D_{x}^{(r)}: \mathbb{C}(x) \rightarrow \mathbb{C}(x)$ stabilize $\mathbb{C}[x]_{n}^{\mathfrak{G}}$, i.e. $D_{x}^{(r)}\left(\mathbb{C}[x]^{\mathfrak{G}_{n}}\right) \subseteq \mathbb{C}[x]^{\mathfrak{S}_{n}}$.
(3) $D_{x}^{(r)}: \mathbb{C}[x]^{\mathfrak{S}_{n}} \rightarrow \mathbb{C}[x]^{\mathfrak{S}_{n}}$ are triangular with respect to the dominance order of $m_{\lambda}(x)$ :

$$
\begin{equation*}
D_{x}^{(r)} m_{\lambda}(x)=\sum_{\mu \leq \lambda} d_{\lambda, \mu}^{(r)} m_{\mu}(x)=d_{\lambda}^{(r)} m_{\lambda}(x)+\sum_{\mu<\lambda} d_{\lambda, \mu}^{(r)} m_{\mu}(x) \quad\left(\lambda \in \mathcal{P}_{n}\right) \tag{4.3}
\end{equation*}
$$

where $d_{\lambda}^{(r)}=e_{r}\left(t^{\delta} q^{\lambda}\right)$ are elementary symmetric functions of $t^{\delta} q^{\lambda}=\left(t^{n-1} q^{\lambda_{1}}, t^{n-2} q^{\lambda_{2}}, \ldots, q^{\lambda_{n}}\right)$.
Theorem A: The $q$-difference operators $D_{x}^{(r)}(r=1, \ldots, n)$ commute with each other:

$$
\begin{equation*}
D_{x}^{(r)} D_{x}^{(s)}=D_{x}^{(s)} D_{x}^{(r)} \quad(r, s=1, \ldots, n), \tag{4.4}
\end{equation*}
$$

Theorem B: Suppose that the parameter $t$ satisfies $t^{k} \notin q^{\mathbb{Z}_{<0}}(k=1, \ldots, n-1)$. Then, for each partition $\lambda \in \mathcal{P}_{n}$ there exists a unique symmetric polynomial $P_{\lambda}(x) \in \mathbb{C}[x]^{\mathfrak{S}_{n}}$ such that
$D_{x}^{(r)} P_{\lambda}(x)=d_{\lambda}^{(r)} P_{\lambda}(x) \quad(r=1, \ldots, n)$
(2) $P_{\lambda}(x)=m_{\lambda}(x)+\sum_{\mu<\lambda} u_{\mu}^{\lambda} m_{\mu}(x) \quad\left(u_{\mu}^{\lambda} \in \mathbb{C}\right)$.

It is convenient to introduce the generation function:

$$
\begin{equation*}
D_{x}(u)=\sum_{r=0}^{n}(-u)^{r} D_{x}^{(r)}=\sum_{I \subseteq\{1, \ldots, n\}}(-u)^{|I|} A_{I}(x) T_{q, x}^{I}, \quad A_{I}(x)=\frac{T_{t, x}^{I}(\Delta(x))}{\Delta(x)} . \tag{4.6}
\end{equation*}
$$

Then the eigenfunction equations for $P_{\lambda}(x)$ is expressed as

$$
\begin{equation*}
D_{x}(u) P_{\lambda}(x)=d_{\lambda}(u) P_{\lambda}(x), \quad d_{\lambda}(u)=\sum_{r=0}^{n}(-u)^{r} e_{r}\left(t^{\delta} q^{\lambda}\right)=\prod_{i=1}^{n}\left(1-u t^{n-i} q^{\lambda_{i}}\right) . \tag{4.7}
\end{equation*}
$$

- $\boldsymbol{d}_{\boldsymbol{\lambda}}^{(r)}=\boldsymbol{e}_{\boldsymbol{r}}\left(\boldsymbol{t}^{\delta} \boldsymbol{q}^{\boldsymbol{\lambda}}\right)$ : For each $I \subseteq\{1, \ldots, n\}$ with $|I|=r$, we have

$$
\begin{equation*}
A_{I}(x)=t^{\binom{r}{2}} \prod_{\substack{i<j \\ i \in I, j \notin I}} t \frac{1-x_{j} / t x_{i}}{1-x_{j} / x_{i}} \prod_{\substack{i<j \\ i \notin I, j \in I}} \frac{1-t x_{j} / x_{i}}{1-x_{j} / x_{i}}=t^{\sum_{i \in I}(n-i)}+\text { (lower order terms), } \tag{4.8}
\end{equation*}
$$

where, for $I=\left\{i_{1}<\cdots<i_{r}\right\}$, the exponent of $t$ is computed as

$$
\begin{equation*}
\binom{r}{2}+|\{(i, j) \mid i<j, i \in I, j \notin I\}|=\binom{r}{2}+\sum_{k=1}^{r}\left(\left(n-i_{k}\right)+(r-k)\right)=\sum_{i \in I}(n-i) . \tag{4.9}
\end{equation*}
$$

Hence, we have

$$
\begin{align*}
D_{x}^{(r)} x^{\mu} & =\sum_{|I|=r} A_{I}(x) q^{\sum_{i \in I} \mu_{i}} x^{\mu}=\left(\sum_{|I|=r} t^{\sum_{i \in I}(n-i)} q^{\sum_{i \in I} \mu_{i}}\right) x^{\mu}+\text { lower order terms }  \tag{4.10}\\
& =e_{r}\left(t^{\delta} q^{\mu}\right) x^{\mu}+(\text { lower order terms })
\end{align*}
$$

This implies

$$
\begin{equation*}
D_{x}^{(r)} m_{\lambda}(x)=e_{r}\left(t^{\delta} q^{\lambda}\right) m_{\lambda}(x)+(\text { lower order terms }) \quad\left(\lambda \in \mathcal{P}_{n}\right) . \tag{4.11}
\end{equation*}
$$

- Condition of genericity: One can show that if $t^{k} \notin q^{\mathbb{Z}_{<0}}(k=1, \ldots, n-1)$, then for any distinct $\lambda, \mu \in \mathcal{P}_{n}, d_{\lambda}(u) \neq d_{\mu}(u)$ as polymomials in $u$, and also for generic $u \in \mathbb{C}$.


### 4.2 Commutativity of the operators $D_{x}^{(r)}(r=1, \ldots, n)$

- Orthogonality implies commutativity: (in the context of Macdonald's book)

One can show that, for each $r=1, \ldots, n, D_{x}^{(r)}$ is self-adjoint with respect to the scalar product defined by $w(x)$, by a similar method we used in the case of $D_{x}=D_{x}^{(1)}$. Since $D_{x}^{(r)}: \mathbb{C}[x]^{\mathfrak{G}_{n}} \rightarrow$ $\mathbb{C}[x]^{\mathfrak{S}_{n}}$ is lower triangular with respect to the dominance order, we have

$$
\begin{equation*}
D_{x}^{(r)} P_{\lambda}(x)=\sum_{\mu \leq \lambda} a_{\lambda, \mu}^{(r)} P_{\mu}(x), \tag{4.12}
\end{equation*}
$$

for some $a_{\lambda, \mu}^{(r)} \in \mathbb{C}$, with leading coefficient $a_{\lambda, \lambda}^{(r)}=d_{\lambda}^{(r)}$. Since

$$
\begin{equation*}
\left\langle D_{x}^{(r)} P_{\lambda}, P_{\mu}\right\rangle=a_{\lambda, \mu}^{(r)}\left\langle P_{\mu}, P_{\mu}\right\rangle, \quad\left\langle P_{\lambda}, D_{x}^{(r)} P_{\mu}\right\rangle=0 \quad(\mu<\lambda), \tag{4.13}
\end{equation*}
$$

and $\left\langle P_{\mu}, P_{\mu}\right\rangle \neq 0$, we have $a_{\lambda, \mu}^{(r)}=0$ for $\mu<\lambda$. This means that $D_{x}^{(r)} P_{\lambda}(x)=d_{\lambda} P_{\lambda}(x)$. In this way, the linear operators $D_{x}^{(r)}: \mathbb{C}[x]^{\mathfrak{S}_{n}} \rightarrow \mathbb{C}[x]^{\mathfrak{S}_{n}}(r=1, \ldots, n)$ are simultaneously diagonalized by the Macdonald basis, and hence commute each other. From this, it follows that $D_{x}^{(r)}(r=1, \ldots, n)$ commute each other as $q$-difference operators, by the following lemma.

Lemma: Let $L_{x}$ be a $q$-difference oprator with rational coefficients, and suppose that $L_{x} f(x)=0$ for all $f(x) \in \mathbb{C}[x]^{\mathfrak{G}_{n}}$. Then $L_{x}=0$ as a $q$-difference operator.

Conversely, suppose we know that $D_{x}^{(r)}$ commutes with $D_{x}=D_{x}^{(1)}$, by some other method. Then for each $P_{\lambda}(x)$, we have

$$
\begin{equation*}
D_{x} D_{x}^{(r)} P_{\lambda}(x)=D_{x}^{(r)} D_{x} P_{\lambda}(x)=D_{x}^{(r)}\left(d_{\lambda} P_{\lambda}(x)\right)=d_{\lambda} D_{x}^{(r)} P_{\lambda}(x) . \tag{4.14}
\end{equation*}
$$

This means that $D_{x}^{(r)} P_{\lambda}(x)$ is also an eigenfunction of $D_{x}$ with eigenvalue $d_{\lambda}$. If $t$ is generic, the eigenspace of $D_{x}$ with eigenvalue $d_{\lambda}$ is one dimensional. Hence, $D^{(r)} P_{\lambda}(x)$ is a constant multiple of $P_{\lambda}(x)$, i.e. $D_{\lambda}^{(r)} P_{\lambda}(x)=$ const. $P_{\lambda}(x)$; the eigenvalue const. must coincides with $d_{\lambda}^{(r)}=e_{r}\left(t^{\delta} q^{\lambda}\right)$ since $D_{x}^{(r)} m_{\lambda}(x)=d_{\lambda}^{(r)}(x) m_{\lambda}(x)+$ (lower order temrs).

- A direct proof of commutativity: (by Ruijsenaars, 1987)

The composition $D_{x}^{(r)} D_{x}^{(s)}$ is computed as

$$
\begin{equation*}
D_{x}^{(r)} D_{x}^{(s)}=\sum_{|I|=r,|J|=s} A_{I}(x) A_{J}\left(q^{\epsilon_{I}} x\right) T_{q, x}^{\epsilon_{I}+\epsilon_{J}} \tag{4.15}
\end{equation*}
$$

where $\epsilon_{I}=\sum_{i \in I} \epsilon_{i}, \epsilon_{i}=\left(\delta_{i, j}\right)_{1 \leq j \leq n} \in \mathbb{Z}^{n}$. Setting $K=I \cap J, L=(I \cup J) \backslash K, P=I \backslash K, Q=J \backslash K$, we rewrite (4.15) as

$$
\begin{equation*}
D_{x}^{(r)} D_{x}^{(s)}=\sum_{\substack{K \cap L=\phi \\|K| \leq \min \{r, s\}}}\left(\sum_{\substack{P \cup Q=L \\|K|+|P|=r,|K|+|Q|=s}} A_{K \cup P}(x) A_{K \cup Q}\left(q^{\epsilon_{K}+\epsilon_{P}} x\right) T_{q, x}^{2 \epsilon_{K}+\epsilon_{L}}\right) . \tag{4.16}
\end{equation*}
$$

Then the commutativity $D_{x}^{(r)} D_{x}^{(s)}=D_{x}^{(s)} D_{x}^{(r)}$ is equivalent to the following statement: For each $K, L \subseteq\{1, \ldots, n\}$ with $K \cap L=\phi$, and for any $p, q \in \mathbb{Z}_{\geq 0}$ such that $p+q=|L|$,

$$
\begin{equation*}
\sum_{\substack{P \cup Q=L \\|P|=P,|Q|=q}} A_{K \cup P}(x) A_{K \cup Q}\left(q^{\epsilon_{K}+\epsilon_{P}} x\right)=\sum_{\substack{P \cup Q=L \\|P|=P,|Q|=q}} A_{K \cup Q}(x) A_{K \cup P}\left(q^{\epsilon_{K}+\epsilon_{Q}} x\right) . \tag{4.17}
\end{equation*}
$$

Analyzing this equality carefully, we see that the statement (4.17) is equivalent to the following.
Lemma: For any $r, s \in \mathbb{Z}_{\geq 0}$ with $r+s=n$,

$$
\begin{equation*}
\sum_{\substack{I \cup J=\{1, \ldots, n\} \\|I|=r,|J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{\left(1-t x_{i} / x_{j}\right)\left(1-q x_{i} / t x_{j}\right)}{\left(1-x_{i} / x_{j}\right)\left(1-q x_{i} / x_{j}\right)}=\sum_{\substack{I \cup J=\{1, \ldots, n\} \\|I|=r,|J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{\left(1-t x_{j} / x_{i}\right)\left(1-q x_{j} / t x_{i}\right)}{\left(1-x_{j} / x_{i}\right)\left(1-q x_{j} / x_{i}\right)} . \tag{4.18}
\end{equation*}
$$

This lemma can be proved by combining the residue calculus of rational functions with the induction on $n$. (In fact, Ruijsenaars proved the commutativity of the elliptic version of $D_{x}^{(r)}(r=1, \ldots, n)$ on the basis of the corresponding identity for the Weierstrass sigma function.)

### 4.3 Determinant representation of $D_{x}(u)$

The generating function $D_{x}(u)$ of the Macdonald-Ruijsenaars $q$-difference operators is expressed by the determinant of a matrix of $q$-difference operators as

$$
\begin{align*}
D_{x}(u) & =\frac{1}{\Delta(x)} \operatorname{det}\left(x_{i}^{n-j}\left(1-u t^{n-j} T_{q, x_{i}}\right)\right)_{i, j=1}^{n} \\
& =\frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_{n}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{n-j}\left(1-u t^{n-j} T_{q, x_{\sigma(j)}}\right) \tag{4.19}
\end{align*}
$$

Note that the product $\prod_{j=1}^{n}$ does not depend on the ordering in this case.

