Introduction to Macdonald polynomials: Lecture 5

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2 Macdonald polynomials: Definition and examples

2.3 Eigenfunctions in the case where n = 2 (continued)

2.4 Macdonald polynomials attached to single rows

(See the summary of Lecture 4.)

3 Orthogonality relations

3.1 Scalar product and orthogonality

We define a meromorphic function w(x) on $(\mathbb{C}^*)^n$ by

$$w(x) = \prod_{1 \le i < j \le n} \frac{(x_i/x_j; q)_{\infty}}{(tx_i/x_j; q)_{\infty}} \frac{(x_j/x_i; q)_{\infty}}{(tx_j/x_i; q)_{\infty}}, \quad (z; q)_{\infty} = \prod_{i=0}^{\infty} (1 - q^i z) \quad (z \in \mathbb{C}, \ |q| < 1).$$
(3.1)

We assume |t| < 1 so that w(x) is holomorphic in a neighborhood of the *n*-dimensional torus

$$\mathbb{T}^{n} = \{ x = (x_{1}, \dots, x_{n}) \in (\mathbb{C}^{*})^{n} \mid |x_{i}| = 1 \ (i = 1, \dots, n) \}.$$
(3.2)

For a pair of holomprhic functions f(x), g(x) in a neighborhood of \mathbb{T}^n , we define the scalar product (symmetric bilinear form) $\langle f, g \rangle$ by

$$\left\langle f,g\right\rangle = \frac{1}{n!} \frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathbb{T}^n} f(x^{-1})g(x)w(x)\frac{dx_1\cdots dx_n}{x_1\cdots x_n} = \frac{1}{n!} \mathrm{CT}\Big[f(x^{-1})g(x)w(x)\Big],\tag{3.3}$$

where CT denotes the *constant term* of the Laurent expansion of a holomorphic function around \mathbb{T}^n . Then the Macdonald polynomials are orthogonal with respect to this scalar product:

$$\langle P_{\lambda}(x), P_{\mu}(x) \rangle = \delta_{\lambda,\mu} N_{\lambda} \quad (\lambda, \mu \in \mathcal{P}_n).$$
 (3.4)

• The constant term and the scalar products are determined explicitly as follow:

$$N_{\phi} = \left(\frac{(t;q)_{\infty}}{(q;q)_{\infty}}\right)^{n} \prod_{i=1}^{n} \frac{(t^{i-1}q;q)_{\infty}}{(t^{i};q)_{\infty}} = \prod_{1 \le i < j \le n} \frac{(t^{j-i};q)_{\infty}(qt^{j-i};q)_{\infty}}{(t^{j-i+1};q)_{\infty}(qt^{j-i-1};q)_{\infty}},$$

$$N_{\lambda} = \prod_{1 \le i < j \le n} \frac{(q^{\lambda_{i}-\lambda_{j}}t^{j-i};q)_{\infty}(q^{\lambda_{i}-\lambda_{j}+1}t^{j-i};q)_{\infty}}{(q^{\lambda_{i}-\lambda_{j}}t^{j-i+1};q)_{\infty}(q^{\lambda_{i}-\lambda_{j}+1}t^{j-i-1};q)_{\infty}}.$$
(3.5)

Research project: How can one derive these explicit formulas? Variations and generalizations? • If $q, t \in \mathbb{R}$ and |q| < 1, |t| < 1, the Macdonald polynomials have real coefficients, and \langle , \rangle defines a positive definite scalar product on $\mathbb{R}[x]^{\mathfrak{S}_n}$.

3.2 Comments on the orthogonality relation

The orthogonality of the Macdonald polynomials is a consequence of the facts that

- (1) The q-difference operator D_x is (formally) self-adjoint w.r.t. the weight function w(x),
- (2) The Macdonald polynomials are separated by the eigenvalues of D_x (for generic t).

• Cauchy's theorem as a basis of q-difference de Rham theory: Let $\varphi(z)$ be a holomorphic function in an neighborhood of a closed curve C in \mathbb{C}^* . Then we have

$$\int_C T_{q,z}(\varphi(z))\frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}, \quad \text{i.e.} \quad \int_C (T_{q,z}-1)(\varphi(z))\frac{dz}{z} = 0 \tag{3.6}$$

if q is sufficiently close to 1 (so that C can be deformed continuously to qC in a domain where $\varphi(z)$ is holomorphic). In particular we have

$$\int_C T_{q,z}(\varphi(z))\psi(z)\frac{dz}{z} = \int_C \varphi(z)T_{q,z}^{-1}(\psi(z))\frac{dz}{z}.$$
(3.7)

This formula play the role of the integration by parts.

• Formal adjoint of a q-difference operator: Let $L \in \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ be a q-difference operator in $x = (x_1, \ldots, x_n)$ with rational coefficients of the form $L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x) T_{q,x}^\mu$ (finite sum), where $T_{q,x}^\mu = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$. We define the formal adjoint L^* of L by $L_x^* = \sum_{\mu \in \mathbb{Z}^\mu} T_{q,x}^{-\mu} a_\mu(x)$, so that $(L_x M_x)^* = M_x^* L_x^*$. Then, we have

$$\int_{\mathbb{T}^n} (L_x f)(x^{-1})g(x)w(x)\frac{dx}{x} = \int_{\mathbb{T}^n} L_{x^{-1}}(f(x^{-1}))g(x)w(x)\frac{dx}{x}$$
$$= \int_{\mathbb{T}^n} f(x^{-1})L_{x^{-1}}^*(g(x)w(x))\frac{dx}{x}$$
$$= \int_{\mathbb{T}^n} f(x^{-1})w(x)^{-1}L_{x^{-1}}^*(w(x)g(x))w(x)\frac{dx}{x}$$
(3.8)

and

$$\langle Lf,g\rangle = \langle f,L^{\dagger}g\rangle, \quad L^{\dagger} = w(x)^{-1}L_{x^{-1}}^{*}w(x),$$
(3.9)

provided that q is sufficiently close to 1 and that Cauchy's theorem can be applied to L_x . We say that L_x is formally self-adjoint with respect to w(x) if $L_x^{\dagger} = L_x$, namely $w(x)L_xw(x)^{-1} = L_{x^{-1}}^*$.

• D_x is formally self-adjoint with respect to w(x): Note that

$$\frac{T_{q,x_i}w(x)}{w(x)} = \prod_{j \neq i} \frac{1 - tx_i/x_j}{1 - x_i/x_j} \prod_{j \neq i} \frac{1 - x_j/qx_i}{1 - tx_j/qx_i} = \frac{A_i(x)}{T_{q,x_i}A_i(x^{-1})} \quad (i = 1, \dots, n).$$
(3.10)

This implies that

$$w(x)D_{x}w(x)^{-1} = \sum_{i=1}^{n} A_{i}(x)\frac{w(x)}{T_{q,x_{i}}w(x)}T_{q,x_{i}} = \sum_{i=1}^{n} T_{q,x_{i}}(A_{i}(x^{-1}))T_{q,x_{i}}$$

$$= \sum_{i=1}^{n} T_{q,x_{i}}A_{i}(x^{-1}) = D_{x^{-1}}^{*}.$$
(3.11)

It can be verified directly that $\langle D_x f, g \rangle = \langle f, D_x g \rangle$ if |t| < |q| < 1. (Note that the poles of $A_i(x)$ along $\Delta(x) = 0$ are canceled by the zeros of w(x).)

• Orthogonality: For $\lambda, \mu \in \mathcal{P}_n$, the equality $\langle D_x P_\lambda(x), P_\mu(x) \rangle = \langle P_\lambda(x), D_x P_\mu(x) \rangle$ implies $d_\lambda \langle P_\lambda, P_\mu \rangle = d_\mu \langle P_\lambda, P_\mu \rangle$. If t is generic, we have $d_\lambda \neq d_\mu$ ($\lambda \neq \mu$), and hence $\langle P_\lambda, P_\mu \rangle = 0$ ($\lambda \neq \mu$).

4 Commuting family of *q*-difference operators

4.1 Macdonald-Ruijsenaars operators of higher order

For each r = 0, 1, ..., n, we define the Macdonald-Ruijsenaars operator of rth order by

$$D_x^{(r)} = \sum_{I \subseteq \{1,\dots,n\}; |I|=r} A_I(x) T_{q,x}^I = \sum_{I \subseteq \{1,\dots,n\}; |I|=r} t^{\binom{r}{2}} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i},$$
(4.1)

where $T_{q,x}^{I} = \prod_{i \in I} T_{q,x_i}$, so that $D_x^{(0)} = 1$, $D_x^{(1)} = D_x$ and $D_n^{(n)} = t^{\binom{n}{2}} T_{q,x_1} \cdots T_{q,x_n}$. Note also that

$$A_{I}(x) = t^{\binom{|I|}{2}} \prod_{i \in I, \, j \notin I} \frac{tx_{i} - x_{j}}{x_{i} - x_{j}} = \frac{T_{t,x}^{I} \Delta(x)}{\Delta(x)}.$$
(4.2)

It is known by Ruijsenaars (1987) that the q-difference operators $D_x^{(r)}$ (r = 1, ..., n) commute with each other. By the same method we applied to D_x , one can directly verify:

(1) $D_x^{(r)}$ (r = 1, ..., n) are invariant under the action of \mathfrak{S}_n . (2) $D_x^{(r)}$: $\mathbb{C}(x) \to \mathbb{C}(x)$ stabilize $\mathbb{C}[x]_n^{\mathfrak{S}}$, i.e. $D_x^{(r)}(\mathbb{C}[x]^{\mathfrak{S}_n}) \subseteq \mathbb{C}[x]^{\mathfrak{S}_n}$.

(3) $D_x^{(r)}: \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ are triangular with respect to the dominance order of $m_{\lambda}(x)$:

$$D_x^{(r)}m_\lambda(x) = \sum_{\mu \le \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x) = d_\lambda^{(r)} m_\lambda(x) + \sum_{\mu < \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x) \qquad (\lambda \in \mathcal{P}_n).$$
(4.3)

where $d_{\lambda}^{(r)} = e_r(t^{\delta}q^{\lambda})$ are elementary symmetric functions of $t^{\delta}q^{\lambda} = (t^{n-1}q^{\lambda_1}, t^{n-2}q^{\lambda_2}, \dots, q^{\lambda_n}).$

Theorem A: The q-difference operators $D_x^{(r)}$ (r = 1, ..., n) commute with each other:

$$D_x^{(r)} D_x^{(s)} = D_x^{(s)} D_x^{(r)} \qquad (r, s = 1, \dots, n),$$
(4.4)

Theorem B: Suppose that the parameter t satisfies $t^k \notin q^{\mathbb{Z}_{<0}}$ (k = 1, ..., n - 1). Then, for each partition $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_{\lambda}(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that

(1)
$$D_x^{(r)} P_{\lambda}(x) = d_{\lambda}^{(r)} P_{\lambda}(x)$$
 $(r = 1, ..., n)$ (2) $P_{\lambda}(x) = m_{\lambda}(x) + \sum_{\mu < \lambda} u_{\mu}^{\lambda} m_{\mu}(x)$ $(u_{\mu}^{\lambda} \in \mathbb{C}).$ (4.5)

It is convenient to introduce the generation function:

$$D_x(u) = \sum_{r=0}^n (-u)^r D_x^{(r)} = \sum_{I \subseteq \{1,\dots,n\}} (-u)^{|I|} A_I(x) T_{q,x}^I, \quad A_I(x) = \frac{T_{t,x}^I(\Delta(x))}{\Delta(x)}.$$
 (4.6)

Then the eigenfunction equations for $P_{\lambda}(x)$ is expressed as

$$D_x(u)P_{\lambda}(x) = d_{\lambda}(u)P_{\lambda}(x), \quad d_{\lambda}(u) = \sum_{r=0}^n (-u)^r e_r(t^{\delta}q^{\lambda}) = \prod_{i=1}^n (1 - ut^{n-i}q^{\lambda_i}).$$
(4.7)

• $d_{\lambda}^{(r)} = e_r(t^{\delta}q^{\lambda})$: For each $I \subseteq \{1, \dots, n\}$ with |I| = r, we have

$$A_{I}(x) = t^{\binom{r}{2}} \prod_{\substack{i < j \\ i \in I, j \notin I}} t \, \frac{1 - x_j/tx_i}{1 - x_j/x_i} \prod_{\substack{i < j \\ i \notin I, j \in I}} \frac{1 - tx_j/x_i}{1 - x_j/x_i} = t^{\sum_{i \in I} (n-i)} + (\text{lower order terms}), \quad (4.8)$$

where, for $I = \{i_1 < \cdots < i_r\}$, the exponent of t is computed as

$$\binom{r}{2} + |\{(i,j) \mid i < j, i \in I, j \notin I\}| = \binom{r}{2} + \sum_{k=1}^{r} ((n-i_k) + (r-k)) = \sum_{i \in I} (n-i).$$
(4.9)

Hence, we have

$$D_{x}^{(r)}x^{\mu} = \sum_{|I|=r} A_{I}(x)q^{\sum_{i\in I}\mu_{i}}x^{\mu} = \Big(\sum_{|I|=r} t^{\sum_{i\in I}(n-i)}q^{\sum_{i\in I}\mu_{i}}\Big)x^{\mu} + \text{lower order terms}$$

$$= e_{r}(t^{\delta}q^{\mu})x^{\mu} + (\text{lower order terms}).$$
(4.10)

This implies

$$D_x^{(r)}m_\lambda(x) = e_r(t^\delta q^\lambda)m_\lambda(x) + (\text{lower order terms}) \qquad (\lambda \in \mathcal{P}_n).$$
(4.11)

• Condition of genericity: One can show that if $t^k \notin q^{\mathbb{Z}_{<0}}$ (k = 1, ..., n - 1), then for any distinct $\lambda, \mu \in \mathcal{P}_n, d_{\lambda}(u) \neq d_{\mu}(u)$ as polynomials in u, and also for generic $u \in \mathbb{C}$.

4.2 Commutativity of the operators $D_x^{(r)}$ (r = 1, ..., n)

• Orthogonality implies commutativity: (in the context of Macdonald's book)

One can show that, for each r = 1, ..., n, $D_x^{(r)}$ is self-adjoint with respect to the scalar product defined by w(x), by a similar method we used in the case of $D_x = D_x^{(1)}$. Since $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is lower triangular with respect to the dominance order, we have

$$D_x^{(r)} P_{\lambda}(x) = \sum_{\mu \le \lambda} a_{\lambda,\mu}^{(r)} P_{\mu}(x), \qquad (4.12)$$

for some $a_{\lambda,\mu}^{(r)} \in \mathbb{C}$, with leading coefficient $a_{\lambda,\lambda}^{(r)} = d_{\lambda}^{(r)}$. Since

$$\left\langle D_x^{(r)} P_\lambda, P_\mu \right\rangle = a_{\lambda,\mu}^{(r)} \left\langle P_\mu, P_\mu \right\rangle, \quad \left\langle P_\lambda, D_x^{(r)} P_\mu \right\rangle = 0 \quad (\mu < \lambda), \tag{4.13}$$

and $\langle P_{\mu}, P_{\mu} \rangle \neq 0$, we have $a_{\lambda,\mu}^{(r)} = 0$ for $\mu < \lambda$. This means that $D_x^{(r)} P_{\lambda}(x) = d_{\lambda} P_{\lambda}(x)$. In this way, the linear operators $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ $(r = 1, \ldots, n)$ are simultaneously diagonalized by the Macdonald basis, and hence commute each other. From this, it follows that $D_x^{(r)}$ $(r = 1, \ldots, n)$ commute each other as q-difference operators, by the following lemma.

Lemma: Let L_x be a q-difference oprator with rational coefficients, and suppose that $L_x f(x) = 0$ for all $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Then $L_x = 0$ as a q-difference operator.

Conversely, suppose we know that $D_x^{(r)}$ commutes with $D_x = D_x^{(1)}$, by some other method. Then for each $P_{\lambda}(x)$, we have

$$D_x D_x^{(r)} P_{\lambda}(x) = D_x^{(r)} D_x P_{\lambda}(x) = D_x^{(r)} (d_{\lambda} P_{\lambda}(x)) = d_{\lambda} D_x^{(r)} P_{\lambda}(x).$$
(4.14)

This means that $D_x^{(r)} P_{\lambda}(x)$ is also an eigenfunction of D_x with eigenvalue d_{λ} . If t is generic, the eigenspace of D_x with eigenvalue d_{λ} is one dimensional. Hence, $D^{(r)} P_{\lambda}(x)$ is a constant multiple of $P_{\lambda}(x)$, i.e. $D_{\lambda}^{(r)} P_{\lambda}(x) = \text{const.} P_{\lambda}(x)$; the eigenvalue const. must coincides with $d_{\lambda}^{(r)} = e_r(t^{\delta}q^{\lambda})$ since $D_x^{(r)} m_{\lambda}(x) = d_{\lambda}^{(r)}(x)m_{\lambda}(x) + (\text{lower order terms}).$

• A direct proof of commutativity: (by Ruijsenaars, 1987)

The composition $D_x^{(r)} D_x^{(s)}$ is computed as

$$D_x^{(r)} D_x^{(s)} = \sum_{|I|=r, |J|=s} A_I(x) A_J(q^{\epsilon_I} x) T_{q,x}^{\epsilon_I + \epsilon_J}$$
(4.15)

where $\epsilon_I = \sum_{i \in I} \epsilon_i$, $\epsilon_i = (\delta_{i,j})_{1 \leq j \leq n} \in \mathbb{Z}^n$. Setting $K = I \cap J$, $L = (I \cup J) \setminus K$, $P = I \setminus K$, $Q = J \setminus K$, we rewrite (4.15) as

$$D_x^{(r)} D_x^{(s)} = \sum_{\substack{K \cap L = \phi \\ |K| \le \min\{r,s\}}} \Big(\sum_{\substack{P \cup Q = L \\ |K| + |P| = r, |K| + |Q| = s}} A_{K \cup P}(x) A_{K \cup Q}(q^{\epsilon_K + \epsilon_P} x) T_{q,x}^{2\epsilon_K + \epsilon_L} \Big).$$
(4.16)

Then the commutativity $D_x^{(r)}D_x^{(s)} = D_x^{(s)}D_x^{(r)}$ is equivalent to the following statement: For each $K, L \subseteq \{1, \ldots, n\}$ with $K \cap L = \phi$, and for any $p, q \in \mathbb{Z}_{\geq 0}$ such that p + q = |L|,

$$\sum_{\substack{P \cup Q = L\\|P| = p, |Q| = q}} A_{K \cup P}(x) A_{K \cup Q}(q^{\epsilon_K + \epsilon_P} x) = \sum_{\substack{P \cup Q = L\\|P| = p, |Q| = q}} A_{K \cup Q}(x) A_{K \cup P}(q^{\epsilon_K + \epsilon_Q} x).$$
(4.17)

Analyzing this equality carefully, we see that the statement (4.17) is equivalent to the following. Lemma: For any $r, s \in \mathbb{Z}_{\geq 0}$ with r + s = n,

$$\sum_{\substack{I \sqcup J = \{1,\dots,n\} \ i \in I \\ |I|=r, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - tx_i/x_j)(1 - qx_i/tx_j)}{(1 - x_i/x_j)(1 - qx_i/x_j)} = \sum_{\substack{I \sqcup J = \{1,\dots,n\} \ i \in I \\ |I|=r, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - tx_j/x_i)(1 - qx_j/tx_i)}{(1 - x_j/x_i)(1 - qx_j/x_i)}.$$
 (4.18)

This lemma can be proved by combining the residue calculus of rational functions with the induction on n. (In fact, Ruijsenaars proved the commutativity of the elliptic version of $D_x^{(r)}$ (r = 1, ..., n)on the basis of the corresponding identity for the Weierstrass sigma function.)

4.3 Determinant representation of $D_x(u)$

The generating function $D_x(u)$ of the Macdonald-Ruijsenaars q-difference operators is expressed by the determinant of a matrix of q-difference operators as

$$D_{x}(u) = \frac{1}{\Delta(x)} \det \left(x_{i}^{n-j} \left(1 - ut^{n-j} T_{q,x_{i}} \right) \right)_{i,j=1}^{n}$$

$$= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_{n}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{n-j} \left(1 - ut^{n-j} T_{q,x_{\sigma(j)}} \right)$$
(4.19)

Note that the product $\prod_{i=1}^{n}$ does not depend on the ordering in this case.