Lecture 5 : 2021/03/05

(II) $n=2$, $\lambda=(\ell)=(\ell,0)$ $\ell=0,1,2,\dots$

$$D_x P_{(\ell)}(x) = d_{(\ell)} P_{(\ell)}(x) \qquad d_{(\ell)} = tq^{\ell}+1$$

$$P_{(\ell)}(x_1,x_2) = x_1^{\ell} \underbrace{\sum_{k=0}^{\ell} \frac{(t;q)_k (q^{-\ell};q)_k}{(q;q)_k (q^{1-\ell}t^{-1};q)_k} (qx_2/tx_1)^k}$$

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a) \quad (k=0,1,2,\dots)$$

q-shifed factorials

$$(q^{-\ell};q)_k = \cdots (1-q^{\underset{\overset{\shortparallel}{0}}{-\ell+\ell}})\cdots$$

$$=0 \qquad \underline{\underline{k>\ell}}$$

$${}_2\phi_1\left(\begin{matrix} t & q^{-\ell} \\ & q^{1-\ell}t^{-1} \end{matrix}; q, qx_2/tx_1\right)$$

$$Q_{(\ell)}(x_1,x_2) = \underbrace{\sum_{\mu_1+\mu_2=\ell} \frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2}}$$ symmetric

$$= \frac{(t;q)_\ell}{(q;q)_\ell} x_1^\ell + (\text{lower order terms})$$

$$= \frac{(t;q)_\ell}{(q;q)_\ell} P_{(\ell)}(x_1,x_2)$$

$(1-q^{\ell-k}t)(1-q^{\ell-k+1}t)\cdots(1-q^{\ell-1}t)$

$\boxed{(1-q^{\ell-k}t)(1-q^{\ell+k}t^{-1})}$

$x_1^\ell (x_2/x_1)^k$

$\mu_1 = \ell-k$

$\mu_2 = k$

$$= \sum_{k=0}^{\ell} \boxed{\frac{(t;q)_{\ell-k}}{(q;q)_{\ell-k}}} \frac{(t;q)_k}{(q;q)_k} x_1^{\ell-k} x_2^k$$

$$\frac{(t;q)_{\ell-k}}{(q;q)_{\ell-k}} = \frac{(t;q)_\ell}{(q;q)_\ell} \frac{(q^{\ell-k+1};q)_k}{(q^{\ell-k}t;q)_k} = \frac{(t;q)_\ell}{(q;q)_\ell} \frac{(q^{-\ell};q)_k}{(q^{1-\ell}t^{-1};q)_k} (qt)^k$$

<u>special limiting case</u>: $t = q^{\beta}$, $q \to 1$ Macdonald $\to$ Jack

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$$

- $(a;q)_k \longrightarrow (1-a)^k \quad (q \to 1)$

- $$\frac{(q^{\alpha};q)_k}{(1-q)^k} = \frac{(1-q^{\alpha})(1-q^{\alpha+1})\cdots(1-q^{\alpha+k-1})}{(1-q)(1-q)\cdots(1-q)}$$

$a = q^{\alpha}$ $\qquad \dfrac{1-q^{\alpha}}{1-q} \longrightarrow \alpha \quad (q \to 1)$ $\qquad q = e^{\varepsilon} \ (\varepsilon \to 0)$

$$\frac{(q^{\alpha};q)_k}{(1-q)^k} \to \alpha(\alpha+1)\cdots(\alpha+k-1) = (\alpha)_k$$

<u>shifted factorial</u>

$$Q_{(\ell)}(x) = \sum_{\mu_1+\mu_2=\ell} \frac{(t:q)_{\mu_1}\,(t:q)_{\mu_2}}{(q:q)_{\mu_1}\,(q:q)_{\mu_2}}\, x_1^{\mu_1} x_2^{\mu_2}$$

$$t = q^{\beta} \quad \downarrow$$

$$Q^{(\beta)}_{(\ell)}(x) = \sum_{\mu_1+\mu_2=\ell} \frac{(\beta)_{\mu_1}\,(\beta)_{\mu_2}}{\mu_1!\;\;\mu_2!}\, x_1^{\mu_1} x_2^{\mu_2}$$

$$(1+z)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k \qquad \binom{\alpha}{k} = \frac{1}{k!}\,\underline{\alpha(\alpha-1)\cdots(\alpha-k+1)}$$

$$(1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}\, z^k \qquad (|z|<1)$$

$$\frac{1}{(1-x_1 y)^\beta\,(1-x_2 y)^\beta} = \left(\sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}\, x_1^k y^k\right)\left(\sum_{k=0}^{\infty} \frac{(\beta)_k}{k!}\, x_2^k y^k\right)$$

$$= \sum_{l=0}^{\infty}\left(\sum_{\mu_1+\mu_2=l} \frac{(\beta)_{\mu_1}}{\mu_1!}\,\frac{(\beta)_{\mu_2}}{\mu_2!}\, x_1^{\mu_1} x_2^{\mu_2}\right) y^l$$

$$= \sum_{l\geq 0}^{\infty} \underline{\underline{Q_{(l)}^{(\beta)}(x_1, x_2)}}\, y^l .$$

Jack polynomial in 2 variables

(11) q-Binomial theorem

$$(z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z) \qquad \underline{|q|<1} \qquad \underline{z\in\mathbb{C}}$$

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k} z^k \qquad \prod_{i=0}^{\infty}(1+a_i) \qquad \underline{\sum_{i=0}^{\infty}|a_i| < +\infty}$$

$$(|z|<1)$$

$$= \sum_{k=0}^{\infty}\frac{(1-a)(1-qa)\cdots(1-q^{k-1}a)}{(1-q)(1-q^2)\cdots(1-q^k)} z^k$$

$$\frac{(q^\alpha z;q)_\infty}{(q^\beta z;q)_\infty} \sim \underbrace{(1-z)^{\beta-\alpha}}_{q\to 1} \qquad (1-z)^{-\alpha} = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(q)_k} z^k$$

<u>Exercise</u> $f(z) = \dfrac{(az:q)_\infty}{(z:q)_\infty} = \sum_{k=0}^{\infty} c_k z^k \qquad (|z|<1)$

$\underline{|z|<1} \qquad z=0$

$f(0) = 1 \quad , \; c_0 = 1$

$$f(qz) = \frac{(qaz:q)_\infty}{(qz:q)_\infty} = \frac{1-z}{1-az}\,\frac{(az:q)_\infty}{(z:q)_\infty} = \frac{1-z}{1-az}\, f(z)$$

$$(1-az)\,f(qz) = (1-z)\,f(z)$$

$\rightsquigarrow$ recurrence relns for $\underline{\underline{c_n}}$

special cases <u>Euler</u> <u>1758</u>

$$\frac{(tx_1y:q)_\infty}{(x_1y:q)_\infty}\,\frac{(tx_2y:q)_\infty}{(x_2y:q)_\infty} = \sum_{\ell=0}^{\infty}\underbrace{\left(\sum_{\mu_1+\mu_2=\ell}\frac{(t:q)_{\mu_1}}{(q:q)_{\mu_1}}\frac{(t:q)_{\mu_2}}{(q:q)_{\mu_2}}x_1^{\mu_1}x_2^{\mu_2}\right)}_{Q_{(\ell)}(x_1,x_2)}y^{\ell}$$

$$\left(\sum_{k\geq 0}^{\infty}\frac{(t:q)_k}{(q:q)_k}x_1^k y^k\right)\left(\qquad\qquad\right)$$

4° Macdonald polynomials $P_{(\ell)}(x)$ $\begin{cases} x=(x_1,\dots x_n) \\ \lambda=(\ell)=(\ell,0,\dots 0)\end{cases}$

$$Q_{(\ell)}(x) = \sum_{\mu_1+\dots+\mu_n=\ell}\frac{(t:q)_{\mu_1}\cdots(t:q)_{\mu_n}}{(q:q)_{\mu_1}\cdots(q:q)_{\mu_n}}x_1^{\mu_1}\cdots x_n^{\mu_n}$$

deg ℓ
symmetric

$$= \frac{(t:q)_\ell}{(q:q)_\ell}x_1^\ell + \dots = \frac{(t:q)_\ell}{(q:q)_\ell}P_{(\ell)}(x) \quad . \blacksquare$$

$$D_x = \sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q, x_i}$$

$$D_x Q_{(\ell)}(x) = d_{(\ell)} Q_{(\ell)}(x)$$

$$d_{(\ell)} = t^{n-1} q^{\ell} + t^{n-2} + \cdots + 1 = t^{n-1} q^{\ell} + \frac{1 - t^{n-1}}{1 - t}$$

$$t^{\delta} q^{\lambda} = (t^{n-1} q^{\ell}, t^{n-2}, \ldots, 1)$$

$$\lambda = (\ell, 0, \ldots, 0)$$

$$\frac{(t x_1 y; q)_\infty \cdots (t x_n y; q)_\infty}{(x_1 y; q)_\infty \cdots (x_n y; q)_\infty} = \sum_{\ell=0}^{\infty} \underline{Q_{(\ell)}(x) y^{\ell}}$$

$x = (x_1, \ldots, x_n)$

$$\overset{\shortparallel}{\Phi}(x; y) \quad y : \text{one}$$

$$\frac{(tx_1y;q)_\infty \cdots (tx_ny;q)_\infty}{(x_1y;q)_\infty \cdots (x_ny;q)_\infty} = \sum_{\ell=0}^{\infty} \underline{Q_{(\ell)}(x)}\, y^\ell \qquad x=(x_1,\dots,x_n)$$

$$\overset{\shortparallel}{\Phi(x;y)} \qquad y : \text{me}$$

$$D_x \Phi(x,y) = \sum_{\ell=0}^{\infty} \underline{D_x(Q_{(\ell)}(x))}\, y^\ell \qquad \underline{t^{n-1}q^\ell + \frac{1-t^{n-1}}{1-t}}$$

$$\left(t^{n-1}\mathcal{T}_{q,y} + \frac{1-t^{n-1}}{1-t}\right)\Phi(x,y) = \sum_{\ell=0}^{\infty} Q_{(\ell)}(x)\, y^\ell \left(t^{n-1}q^\ell + \frac{1-t^{n-1}}{1-t}\right)$$

$$D_x \Phi(x,y) = \left(t^{n-1}\mathcal{T}_{q,y} + \frac{1-t^{n-1}}{1-t}\right)\Phi(x,y).$$

$$D_x \Phi(x,y) = \left(t^{n-1} T_{q,y} + \frac{1-t^{n-1}}{1-t} \right) \Phi(x,y) . \quad (*)$$

$$\Phi(x,y) = \prod_{j=1}^{n} \frac{(t x_j y ; q)_\infty}{(x_j y ; q)_\infty}$$

Kernel function identity

$$T_{q,x_i} \Phi(x,y) = \frac{1 - x_i y}{1 - t x_i y} \Phi(x,y) \qquad \underline{D_x(1) = d_0 \cdot 1} \quad \begin{pmatrix} y = 0 \\ y \to \infty \end{pmatrix}$$

$$T_{q,y} \Phi(x,y) = \prod_{j=1}^{n} \frac{1 - x_j y}{1 - t x_j y} \Phi(x,y) \qquad \frac{\text{degree } n}{\text{degree } n}$$

Source identity

$$(*) \Leftrightarrow \sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \times \frac{1 - x_i y}{1 - t x_i y} = t^{n-1} \prod_{j=1}^{n} \frac{1 - x_j y}{1 - t x_j y} + \frac{1 - t^{n-1}}{1 - t} .$$

$$\Phi(x;y) = \frac{(tx_1y;q)_\infty (tx_2y;q)_\infty \cdots (tx_ny;q)_\infty}{(x_1y;q)_\infty (x_2y;q)_\infty \cdots (x_ny;q)_\infty} = \sum_{\ell=0}^{\infty} \underbrace{Q_{(\ell)}(x)}_{\frac{(t;q)_\ell}{(q;q)_\ell} P_{(\ell)}(x)} y^\ell$$

$x = (x_1, \dots x_n)$

$$x = t^\delta = (t^{n-1}, t^{n-2}, \dots, 1)$$

$$\Phi(t^\delta;y) = \frac{(t^n y;q)_\infty \cancel{(t^{n-1}y;q)_\infty} \cdots \cancel{(ty;q)_\infty}}{\cancel{(t^{n-1}y;q)_\infty} \cancel{(t^{n-2}y;q)_\infty} \cdots \cancel{(y;q)_\infty}}$$

$$= \frac{(t^n y;q)_\infty}{(y;q)_\infty} = \sum_{\ell=0}^{\infty} \frac{(t^n;q)_\ell}{(q;q)_\ell} y^\ell$$

$$\frac{(t;q)_\ell}{(q;q)_\ell} P_\ell(t^\delta) = \frac{(t^n;q)_\ell}{(q;q)_\ell} \qquad P_\ell(t^\delta) = \frac{(t^n;q)_\ell}{(t;q)_\ell}$$

$\blacksquare$

- $n=2$

$$Q_{(\ell)}(x) = \frac{(t;q)_\ell}{(q;q)_\ell} x_1^\ell \, {}_2\phi_1\left(\begin{matrix} t & q^{-\ell} \\ & q^{1-\ell}t^{-1} \end{matrix} ; q, q x_2/t x_1 \right)$$

- n:

$$\phi_D\left(\begin{matrix} a; b_1 \cdots b_m \\ c \end{matrix} ; q; z_1, \dots, z_m \right)$$

F_D: Lauricella

(F_1: Appell)

$$= \sum_{k_1, \dots, k_m \geq 0} \frac{(a;q)_{k_1+\dots+k_m} (b_1;q)_{k_1} \cdots (b_m;q)_{k_m}}{(c;q)_{k_1+\dots+k_m} (q;q)_{k_1} \cdots (q;q)_{k_m}} z_1^{k_1} \cdots z_m^{k_m}$$

$$Q_{(\ell)}(x) = \frac{(t;q)_\ell}{(q;q)_\ell} x_1^\ell \, \phi_D\left(\begin{matrix} q^{-\ell}; t, \dots, t \\ q^{1-\ell}t^{-1} \end{matrix} ; q; \overbrace{q x_2/t x_1, \dots, q x_n/t x_1}^{n-1} \right)$$

§3: Orthogonality relations

1°: Scalar product & orthogality

$x = (x_1, \dots x_n)$ coordinates of $(\mathbb{C}^*)^n$ algebraic torus

$\mathbb{T}^n = (S^1)^n$

$$w(x) = w_+(x)\, w_+(x^{-1}) \qquad \underline{\underline{|\Delta(x)|^{2\beta}}} \qquad t = q^\beta$$

$$w_+(x) = \prod_{1 \le i < j \le n} \frac{(x_i/x_j : q)_\infty}{(t x_i/x_j : q)_\infty} \sim \prod_{1 \le i < j \le n} (1 - x_i/x_j)^\beta$$

f, g: holomorphic function in a neighborhood of $\mathbb{T}^n$ $\begin{cases} \mathbb{T}^n = \{ |x_1| = \dots = |x_n| = 1 \} \\ |t| < 1 \end{cases}$

$$\langle f, g \rangle = \frac{1}{n!} \frac{1}{(2\pi\sqrt{-1})^n} \int_{\mathbb{T}^n} f(x^{-1}) g(x) w(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

$$= \frac{1}{n!} \underbrace{CT}_{\text{constant term.}} \left[f(x^{-1}) g(x) w(x) \right]$$

<u>Theorem</u> $\langle P_\lambda(x), P_\mu(x) \rangle = \delta_{\lambda\mu} \underline{\underline{N_\lambda}} \quad (\lambda, \mu \in \mathcal{P}_n)$

<u>Remark</u>

$$\begin{cases} N_\lambda = \displaystyle\prod_{1 \le i < j \le n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i} ; q)_\infty (q^{\lambda_i - \lambda_j + 1} t^{j-i} ; q)_\infty}{(q^{\lambda_i - \lambda_j} t^{j-i+1} ; q)_\infty (q^{\lambda_i - \lambda_j + 1} t^{j-i-1} ; q)_\infty} \\ N_\phi = \frac{1}{n!} [w(x)] = \left(\frac{(t;q)_\infty}{(q;q)_\infty} \right)^n \displaystyle\prod_{i=1}^{n} \frac{(t^{i-1} q ; q)_\infty}{(t^i ; q)_\infty} \end{cases}$$

$\leftarrow$

(iii) Comments on orthogonality

(1) D_x : "<u>self-adjoint</u>" w.r.t. $w(x)$

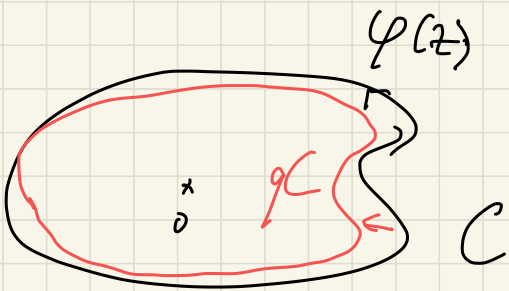
(2) $P_\lambda(x)$: are separated by the eigenvalues $\underline{\underline{d_\lambda}}$

(generic q, t) $\qquad \lambda, \mu \in \mathbb{P}_n \quad \lambda \neq \mu \Rightarrow d_\lambda \neq d_\mu$.

- <u>Cauchy's theorem</u> (<u>integration by parts</u>)

$C \subseteq \mathbb{C}^*$ closed curve

$\varphi(z)$: holomorphic in a neighborhood of C



$qC \sim C$ $\qquad$ q : sufficiently close to 1

$$\int_C \underbrace{T_{q,z}\varphi(z)}_{\varphi(qz)} \frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}$$

$$\parallel$$

$$\int_{qC} \varphi(z)\frac{dz}{z}$$

$$\int_C (T_{q,z}-1)\varphi(z)\frac{dz}{z} = 0$$

$$\int_C (T_{q,z}\varphi(z))\,\psi(z)\frac{dz}{z} = \int_C \varphi(z)\,T_{q,z}^{-1}(\psi(z))\frac{dz}{z}$$

- $$L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x)\, T_{q,x}^{\mu} \qquad T_{q,x}^{\mu} = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$$

(finite sum)

$$\in \underset{\sim}{\mathbb{C}(x)}\left[T_{q,x}^{\pm 1}\right]$$

$$L_x^* = \sum_{\mu \in \mathbb{Z}^n} T_{q,x}^{-\mu}\, a_\mu(x) \qquad (L_x M_x)^* = M_x^* L_x^*$$

formal adjoint

$$\langle \underline{L} f, g \rangle = \langle f, L^{\dagger} g \rangle$$

$L^{\dagger}$: formal adjoint w.r.t. $w(x)$

$$\langle L_x f, g\rangle \propto \int_{\mathbb{T}^n} (L_x f)(x^{-1})\, g(x)\, w(x) \frac{dx}{x}$$

$$= \int_{\mathbb{T}^n} L_{x^{-1}}\big(f(x^{-1})\big)\, g(x)\, w(x) \frac{dx}{x}$$

$$= \int_{\mathbb{T}^n} f(x^{-1})\, L^{*}_{x^{-1}}\big(g(x)\, w(x)\big) \frac{dx}{x}$$

$$= \int_{\mathbb{T}^n} f(x^{-1})\, \underbrace{w(x)^{-1} L^{*}_{x^{-1}}\big(w(x) g(x)\big)}\, w(x) \frac{dx}{x}$$

$$\propto \langle f, L^{\dagger}_x g\rangle \qquad \underbrace{L^{\dagger}_x = w(x)^{-1} L^{*}_{x^{-1}}\, w(x)}$$

- D_x is formally self-adjoint w.r.t. $w(x)$

$$D_x = w(x)^{-1}\left(D_{x^{-1}}\right)^{*} w(x)$$

$$\underline{\underline{w(x)\,D_x = \left(D_{x^{-1}}\right)^{*} w(x)}}$$

$$\underline{\langle D_x f(x), g(x)\rangle = \langle f(x), D_x g(x)\rangle}$$

"formally"

$\mathbb{P}^n$

$$D_x = \sum_{i=1}^{n} \prod_{j\neq i} \frac{t_{x_i - x_j}}{\underline{x_i - x_j}}\, T_{q,x_i}$$

$$\prod_{i<j}\left(1 - x_i/x_j\right)$$

$$\langle D_x P_\lambda, P_\mu \rangle = \langle P_\lambda, D_x P_\mu \rangle$$

$$\| \qquad\qquad\qquad \|$$

$$d_\lambda \langle P_\lambda, P_\mu \rangle \qquad\qquad d_\mu \langle P_\lambda, P_\mu \rangle$$

$$\lambda \neq \mu \Rightarrow d_\lambda \neq d_\mu \Rightarrow \langle P_\lambda, P_\mu \rangle = 0 \quad \blacksquare$$

<u>Remark</u> $\langle f, g \rangle = \cdots \int f(x^{-1}) g(x) w(x) \frac{dx}{x}$

$\cdot\ q, t \in \mathbb{R} \quad |q|<1, |t|<1 \qquad \underset{\sim}{P_\lambda}(x) \in \mathbb{R}[x] \qquad |w_t(x)|^2$

$$\langle f, g \rangle = \int \overline{f(x)}\, g(x)\, \underline{w(x)} \frac{dx}{x}$$

$$D_x^{(r)} = \sum_{\substack{I \subseteq \{1 \dots n\} \\ |I| = r}} A_I(x) \, T_{q,x}^{I} \qquad A_I(x) = t^{\binom{|I|}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j}$$

$$T_{q,x}^{I} = \prod_{i \in I} T_{q,x_i}$$

$$\begin{cases} (A) \quad [D_x^{(r)}, D_x^{(s)}] = 0 \qquad (r, s \in \{1 \dots n\}) \\ (B) \quad \underline{D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x)} \qquad (\lambda \in \mathbb{P}_n) \end{cases}$$

$$d_\lambda^{(r)} = e_r(t^\delta q^\mu)$$

q, t : generic

$$\underline{\underline{D_x^{(r)}}} \; ; \; D_x = D_x^{(1)}$$