Introduction to Macdonald polynomials: Lecture 6

by M. Noumi [March 12, 2021]

4 Commuting family of q-difference operators

4.3 Determinant representation of $D_x(u)$

The generating function $D_x(u)$ of the Macdonald-Ruijsenaars q-difference operators is expressed in terms of the determinant of a matrix of q-difference operators:

$$\begin{aligned} D_x(u) &= \frac{1}{\Delta(x)} \det\Big(x_i^{n-j}\big(1-ut^{n-j}T_{q,x_i}\big)\Big)_{i,j=1}^{n} \\ &= \frac{1}{\Delta(x)} \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{n-j}\big(1-ut^{n-j}T_{q,x_{\sigma(j)}}\big). \end{aligned} \tag{4.1}$$

The q-difference operators $L_{ij} = x_i^{n-j}(1-ut^{n-j}T_{q,x_i})$ satisfy the commutativity $L_{i,j}L_{k,l} = L_{k,l}L_{i,j}$ $(i \neq k)$. This implies that the product $\prod_{i=1}^{n}$ above does not depend on the ordering:

For a q-dfference operator $L_x = \sum_{\mu\in\mathbb{Z}^n} a_\mu(x) T_{q,x}^{\mu} \in \mathbb{C}(x)[T_{q,x}^{\pm 1}]$, we define its *symbol* by

$$\mathrm{symb}(L_x) = \sum_{\mu\in\mathbb{Z}^n} a_\mu(x)\xi^\mu \in \mathbb{C}(x)[\xi^{\pm 1}], \quad \xi = (\xi_1,\ldots,\xi_n). \tag{4.2}$$

Note that two q-difference operators L_x, M_x coincide if $\mathrm{symb}(L_x) = \mathrm{symb}(M_x)$. We compute the symbol of $D_x(u)$ as follows:

$$\begin{aligned} \mathrm{symb}(D_x(u)) &= \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} \frac{T_{t,x}^{\epsilon_I}\Delta(x)}{\Delta(x)}\xi^{\epsilon_I} = \frac{1}{\Delta(x)}\Big(\sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|}\xi^{\epsilon_I}T_{t,x}^{\epsilon_I}\Big)\Delta(x) \\ &= \frac{1}{\Delta(x)}\prod_{i=1}^{n}(1-u\,\xi_i\,T_{t,x_i})\Delta(x) = \frac{1}{\Delta(x)}\det\big(x_i^{n-j}(1-u\,t^{n-j}\xi_i)\big)_{i,j=1}^{n} \\ &= \frac{1}{\Delta(x)}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\prod_{j=1}^{n} x_{\sigma(j)}^{n-j}(1-u\,t^{n-j}\xi_{\sigma(j)}), \end{aligned} \tag{4.3}$$

which coincides with the symbol of the right-hand side of (4.1).

- **Limit to the differential (Jack) case:** Noting that

$$q = e^{\varepsilon} \implies T_{q,x_i}x^\mu = q^{\mu_i}x^\mu = \sum_{k=0}^{\infty}\frac{(\mu_i\varepsilon)^k}{k!}x^\mu = \sum_{k=0}^{\infty}\frac{(\varepsilon x_i\partial_{x_i})^k}{k!}x^\mu = e^{\varepsilon x_i\partial_{x_i}}x^\mu = q^{x_i\partial_{x_i}}x^\mu, \tag{4.4}$$

we rewrite the q-shift operators as $T_{q,x_i} = q^{x_i\partial_{x_i}}$ by the Euler operators $x_i\partial_{x_i}$ $(i=1,\ldots,n)$. Then we take the scaling limit of $D_x(u)/(1-q)^n$ as $q\to 1$ with $t=q^\beta$, $u=q^v$:

$$\begin{aligned} L_x(v) &= \lim_{q\to 1}\frac{1}{(1-q)^n}\big(D_x(q^v)\big|_{t=q^\beta}\big) = \frac{1}{\Delta(x)}\lim_{q\to 1}\det\Big(x_i^{n-j}\,\frac{1-q^{v+(n-j)\beta+x_i\partial_{x_i}}}{1-q}\Big)_{i,j=1}^n. \\ &= \frac{1}{\Delta(x)}\det\Big(x_i^{n-j}(v+x_i\partial_{x_i}+(n-j)\beta)\Big)_{i,j=1}^n. \end{aligned} \tag{4.5}$$

The resulting opertor $L_x(v)$ is the generating function of a commuting family of differential operators, called the *Sekiguchi operators*, such that

$$L_x(v)P_\lambda^{(\beta)}(x) = P_\lambda^{(\beta)}(x)\prod_{i=1}^n(v+\lambda_i+(n-i)\beta) \qquad (\lambda\in\mathcal{P}_n), \tag{4.6}$$

where $P_\lambda^{(\beta)}(x) = \lim_{q\to 1}P_\lambda(x;q,q^\beta)$ stand for the Jack polynomials.

Remark: In the section on "Affine Hecke algebras and Dunkl operators", we will explain a construction of the q-difference operators $D_x^{(r)}$ as well as their commutativity, following the idea of Cherednik based on a representation of the affine Hecke algebra.

5 Self-duality and Pieri formula

5.1 Evaluation, self-duality, Pieri formula

• **Evaluation at $x = t^\delta$ (principal specilization):** The hook-length formula for $s_\lambda(t^\delta)$ (Lecture 2, (1.17)) is generalized as follows to the case of Macdonald polynomials.

$$P_\lambda(t^\delta) = t^{n(\lambda)}\prod_{s\in\lambda}\frac{1-t^{n-l'_\lambda(s)}q^{a_\lambda(s)}}{1-t^{l_\lambda(s)+1}q^{a_\lambda(s)}} = \frac{t^{n(\lambda)}\prod_{i=1}^n(t^{n-i};q)_{\lambda_i}}{\prod_{1\le i\le j\le n}(t^{j-i+1}q^{\lambda_i-\lambda_j};q)_{\lambda_j-\lambda_{j+1}}} \tag{5.1}$$

where $n(\lambda)=\sum_{i=1}^n(i-1)\lambda_i$ and, for each $s=(i,j)\in\lambda$, $l'_\lambda(s)=i-1$ denotes the *co-leg length.*

• **Self-duality (evaluation symmetry):** We normalize the Macdonald polynomials as

$$\widetilde{P}_\lambda(x) = \frac{P_\lambda(x)}{P_\lambda(t^\delta)} \qquad (\lambda\in\mathcal{P}_n) \tag{5.2}$$

so that $\widetilde{P}_\lambda(t^\delta)=1$. Then we have the evaluation symmetry (self-duality)

$$\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda) \qquad (\lambda,\mu\in\mathcal{P}_n). \tag{5.3}$$

• **Pieri formula:** For each $r=1,\ldots,n$, we have

$$e_r(x)P_\mu(x) = \sum_{\lambda/\mu:\,\text{v.s},\ |\lambda/\mu|=r}\psi'_{\lambda/\mu}\,P_\lambda(x) \qquad (\mu\in\mathcal{P}_n), \tag{5.4}$$

where the sum is over all partitions $\lambda\in\mathcal{P}_n$ with $\mu\subseteq\lambda$, $|\lambda|=|\mu|+r$, such that the skew diagram λ/μ is a *vertical strip* (i.e. the complement $\lambda\backslash\mu$ contains at most one box in each row). The expansion coefficients are determined as $\psi'_{\lambda/\mu}=\psi_{\lambda'/\mu'}(t,q)$, where

$$\psi_{\lambda/\mu}=\psi_{\lambda/\mu}(q,t)=\prod_{1\le i\le j\le \ell(\mu)}\frac{(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\lambda_i-\mu_i}}{(t^{j-i}q^{\mu_i-\mu_j+1};q)_{\lambda_i-\mu_i}}\frac{(t^{j-i}q^{\mu_i-\lambda_{j+1}+1};q)_{\lambda_i-\mu_i}}{(t^{j-i+1}q^{\mu_i-\lambda_{j+1}};q)_{\lambda_i-\mu_i}}. \tag{5.5}$$

5.2 Self-duality implies the Pieri formula

Note that the fact that $P_\lambda(t^\delta) \neq 0$ follows from the evaluation (principal specialization) of the spacial case $t = q$ where $P_\lambda(x|q,q) = s_\lambda(x)$. Assuming that the self-duality (5.3) has been established, we explain here how one can obtain the Pieri formula (5.4) and the evaluation formula (5.1) from the q-difference equations for $P_\lambda(x)$, by way of the self-duality.

For each $r = 1, \ldots, n$, the eigenfunction equation

$$D_x^{(r)} \widetilde{P}_\lambda(x) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(x) \tag{5.6}$$

implies

$$\sum_{|I|=r} A_I(x) \widetilde{P}_\lambda(q^{\epsilon_I} x) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(x), \tag{5.7}$$

where $\epsilon_I = \sum_{i \in I} \epsilon_i$. Evaluating this formula at $x = t^\delta q^\mu$ ($\mu \in \mathcal{P}_n$), we obtain

$$\sum_{|I|=r} A_I(t^\delta q^\mu) \widetilde{P}_\lambda(t^\delta q^{\mu+\epsilon_I}) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu). \tag{5.8}$$

Suppose that $\nu = \mu + \epsilon_I$ is *not* a partition, i.e. $\mu_{i-1} = \mu_i$ for some $i \in \{2, \ldots, n\}$ with $i \in I$ and $i - 1 \notin I$. In such a case, we have

$$A_I(t^\delta q^\mu) = t^{\binom{|I|}{2}} \prod_{i \in I,\, j \notin J} \frac{t^{n-i+1} q^{\mu_i} - t^{n-j} q^{\mu_j}}{t^{n-i} q^{\mu_i} - t^{n-j} q^{\mu_j}} = 0 \tag{5.9}$$

since $t^{n-i+1} q^{\mu_i} - t^{n-i+1} q^{\mu_{i-1}} = 0$ ($i \in I, j = i-1 \notin I$). This means that the sum in the left-hand side of (5.7) is over all $I \subseteq \{1, \ldots, n\}$ with $|I| = r$ such that $\nu = \mu + \epsilon_I$ is a partition. A skew partition ν/μ is called a *vertical strip* if $\nu = \mu + \epsilon_I$ for sum $I \subseteq \{1, \ldots, n\}$. Namely,

$$\sum_{\nu/\mu:\text{v.s.}\, |\nu/\mu|=r} A_{\nu-\mu}(t^\delta q^\mu) \widetilde{P}_\lambda(t^\delta q^\nu) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu). \tag{5.10}$$

We now apply the symmetry (5.3) to obtain

$$\sum_{\nu/\mu:\text{v.s.}\, |\nu/\mu|=r} A_{\nu-\mu}(t^\delta q^\mu) \widetilde{P}_\nu(t^\delta q^\lambda) = e_r(t^\delta q^\lambda) \widetilde{P}_\mu(t^\delta q^\lambda). \tag{5.11}$$

This means that equality

$$e_r(x) \widetilde{P}_\mu(x) = \sum_{\nu/\mu:\text{v.s.}\, |\nu/\mu|=r} B_{\nu/\mu} \widetilde{P}_\nu(x), \quad B_{\nu/\mu} = A_{\nu-\mu}(t^\delta q^\mu) \tag{5.12}$$

holds for $x = t^\delta q^\lambda$ ($\lambda \in \mathcal{P}_n$); it also implies that (5.12) is an identiy in the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials. Namely, if the self-duality (5.3) has benn established, the q-difference equations (5.6) for $\lambda \in \mathcal{P}_n$ implies the *Pieri formulas* (5.12) for the normalized Macdonald polynomials $\widetilde{P}_\mu(x)$.

5.3 Evaluation at $x = t^\delta$

The normalized Macdonald polynomials $\widetilde{P}_\lambda(x)$ can be written as

$$\widetilde{P}_\lambda(x) = \frac{1}{a_\lambda} P_\lambda(x) = \frac{1}{a_\lambda} m_\lambda(x) + (\text{lower oreder terms}), \quad a_\lambda = P_\lambda(t^\delta). \tag{5.13}$$

Comparing the coefficients of $m_{\mu+(1^r)}(x)$ of the both sides of (5.12), we obtain

$$\frac{1}{a_\mu} = B^{(r)}_{(\mu+(1^r))/\mu} \frac{1}{a_{\mu+(1^r)}}, \quad \text{i.e.} \quad \frac{a_{\mu+(1^r)}}{a_\mu} = B^{(r)}_{(\mu+(1^r))/\mu} = A_{\{1,\ldots,r\}}(t^\delta q^\mu). \tag{5.14}$$

Applying this recurrence formula repeatedly, we obtain the explicit formula (5.1) for $a_\lambda = P_\lambda(t^\delta)$ for arbitrary $\lambda \in \mathcal{P}_n$.

As to the Pieri coefficients $\psi'_{\lambda/\mu}$, from (5.12) we have

$$\psi'_{\lambda/\mu} = \frac{a_\mu}{a_\lambda} B_{\lambda/\mu}, \quad B_{\lambda/\mu} = A_{\lambda-\mu}(t^\delta q^\mu). \tag{5.15}$$

Writing down this formula in terms of $\lambda, \mu \in \mathcal{P}_n$, we obtain the explicit formula for $\psi'_{\lambda/\mu} = \psi_{\lambda'/\mu'}(t,q)$ as in (5.5).

5.4 Comments on the proof of self-dualiy

• **Koornwinder's proof:** One can prove the following two statements for $\mu \in \mathcal{P}_n$ simultaneously by the induction on $|\mu|$ combined with the dominance order (see Macdonald [1]):

$$\begin{aligned} &(a) \qquad \widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda) \quad (\lambda \in \mathcal{P}_n) \\ &(b) \qquad e_r(x)\widetilde{P}_\kappa(x) = \sum_\nu B_{\nu/\kappa}\, \widetilde{P}_\nu(x) \quad (r = 1, \ldots, n; \kappa \in \mathcal{P}_n, \kappa + (1^r) \le \mu) \end{aligned} \tag{5.16}$$

• **Double affine Hecke algebras:** The self-duality can be proved by means of the *Cherednik involution* of the double affine Hecke algebra (to be explained later).

6 Kernel functions (generating functions)

6.1 Kernel functions of Cauchy type and of dual Cauchy type:

• **Kernel function of Cauchy type:** For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we have

$$\Pi(x;y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_{\ell(\lambda) \le \min\{m,n\}} b_\lambda P_\lambda(x) P_\lambda(y), \tag{6.1}$$

where λ runs over all partitions with $\ell(\lambda) \le \min\{m,n\}$, and the coefficients b_λ are determined as

$$b_\lambda = \prod_{s\in\lambda} \frac{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}} = \prod_{1 \le i \le j \le \ell(\lambda)} \frac{(t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}{(t^{j-i} q^{\lambda_i - \lambda_j + 1}; q)_{\lambda_j - \lambda_{j+1}}}. \tag{6.2}$$

- **Kernel function of dual Cauchy type:** For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we have

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j) = \sum_{\ell(\lambda)\subseteq(n^m)} P_\lambda(x|q,t)P_{\lambda'}(y|t,q), \tag{6.3}$$

where the sum is over all partitions λ contained in the $m \times n$ rectangle $(n^m) = (n, \ldots, n)$.

6.2 Kernel function identities and source identities

We consider the case where $m = n$. We first remark that there exists an expansion formula as (6.1) with *some* constants b_λ, if and only if $\Pi(x; y)$ satisfies the *kernel function identity*

$$D_x(u)\Pi(x; y) = D_y(u)\Pi(x; y). \tag{6.4}$$

Expand $\Pi(x; y)$ in terms of Macdonald polynomials $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ as

$$\Pi(x; y) = \sum_{\lambda\in\mathcal{P}_n} P_\lambda(x)Q_\lambda(y), \tag{6.5}$$

Since

$$D_x(u)\Pi(x; y) = \sum_{\lambda\in\mathcal{P}_n} P_\lambda(x)Q_\lambda(y)\prod_{i=1}^{n}(1 - ut^{n-j}q^{\lambda_j}), \tag{6.6}$$

identity (6.4) implies $D_y(u)Q_\lambda(y) = Q_\lambda(y)\prod_{i=1}^{n}(1 - ut^{n-j}q^{\lambda_j})$ and hence, $Q_\lambda(y)$ is a constant multiple of $P_\lambda(y)$.

We now prove (6.4). Since

$$T_{q,x_i}\Pi(x; y) = \prod_{l=1}^{n}\frac{1 - x_i y_l}{1 - t x_i y_l}\,\Pi(x; y), \qquad T_{q,y_k}\Pi(x; y) = \prod_{j=1}^{n}\frac{1 - x_j y_k}{1 - t x_j y_k}\,\Pi(x; y), \tag{6.7}$$

(6.4) is equivalent to the *source identity*

$$\begin{aligned}&\sum_{I\subseteq\{1,\ldots,n\}}(-u)^{|I|}t^{\binom{|I|}{2}}\prod_{i\in I;\, j\notin I}\frac{tx_i - x_j}{x_i - x_j}\prod_{i\in I}\prod_{l=1}^{n}\frac{1 - x_i y_l}{1 - t x_i y_l}\\ &= \sum_{K\subseteq\{1,\ldots,n\}}(-u)^{|K|}t^{\binom{|K|}{2}}\prod_{k\in K;\, l\notin K}\frac{ty_k - y_l}{y_k - y_l}\prod_{k\in K}\prod_{j=1}^{n}\frac{1 - x_j y_k}{1 - t x_j y_k}.\end{aligned} \tag{6.8}$$

An important observations is that this identity does *not* involve q. This means that, in order to prove (6.8), it is sufficient to prove (6.4) for $q = t$. However, we already know that (6.4) holds when $q = t$ by the Cauchy formula for Schur functions.

The existence of an expansion formula of the form (6.1) for difference number of variables m, n follows from the fact that

$$P_\lambda(x_1, \ldots, x_{n-1}, 0) = \begin{cases} P_\lambda(x_1, \ldots, x_{n-1}) & (\ell(\lambda) < n)\\ 0 & (\ell(\lambda) = n).\end{cases} \tag{6.9}$$

It should be noted that we need some other arguments to obtain the explicit formula (6.2) for b_λ. We also remark that (6.1) for the case where $m \geq n$ corresponds to the kernel function identity

$$D_x(u)\Pi(x; y) = (u; t)_{m-n}D_y(ut^{m-n})\Pi(x; y). \tag{6.10}$$