Introduction to Macdonald polynomials: Lecture 7

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- **Corretion of formula (5.1) in Lecture 6:**

$$P_\lambda(t^\delta) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-t^{n-l'_\lambda(s)}q^{a'_\lambda(s)}}{1-t^{l_\lambda(s)+1}q^{a_\lambda(s)}} = \frac{t^{n(\lambda)}\prod_{i=1}^{n}(t^{n-i+1};q)_{\lambda_i}}{\prod_{1\le i\le j\le n}(t^{j-i+1}q^{\lambda_i-\lambda_j};q)_{\lambda_j-\lambda_{j+1}}} \tag{5.1}$$

where $n(\lambda)=\sum_{i=1}^{n}(i-1)\lambda_i$ and, for each $s=(i,j)\in\lambda$, $l'_\lambda(s)=i-1$ and $a'_\lambda(s)=j-1$ denote the *co-leg length* and the *co-arm length*, respectively.

7 Pieri coefficients and branching coefficients

7.1 LR coefficients and branching coefficients

For a pair of partitions $\mu,\nu\in\mathcal{P}_n$, we consider the expand the product $P_\mu(x)P_\nu(x)$ of Macdonalds polynomials in the form

$$P_\mu(x)P_\nu(x) = \sum_{\lambda\in\mathcal{P}_n;\ |\lambda|=|\mu|+|\nu|} c^{\lambda}_{\mu,\nu}\, P_\lambda(x). \tag{7.1}$$

The expansion coefficients $c^\lambda_{\mu,\nu}$ are called the *Littlewood-Richardson coefficients* (or *Clebsch-Gordan coefficients*). If $\nu=(1^r)$ is a single column ($r=0,1,\ldots,n$), the LR coefficients $c^\lambda_{\mu,(1^r)}$ are nothing but the Pieri coefficients $\psi'_{\lambda/\mu}=\psi_{\lambda'/\mu'}(t,q)$ we discussed above since $P_{(1^r)}(x)=e_r(x)$:

$$P_\mu(x)e_r(x) = \sum_{\lambda\in\mathcal{P}_n;\ |\lambda|=|\mu|+r} \psi'_{\lambda/\mu}\, P_\lambda(x) \quad (r=0,1,\ldots,n), \tag{7.2}$$

where the sum is over the *vertical strips* λ/μ with $|\lambda/\mu|=r$.

These Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$ are closely related to the *branching coefficients* $b^\lambda_{\mu,\nu}$ to be defined below. We consider to expand the Macdonald polynomials $P_\lambda(x,y)$ ($\lambda\in\mathcal{P}_{m+n}$) in $m+n$ variables $(x,y)=(x_1,\ldots,x_m,y_1,\ldots,y_n)$ in terms of the Macdonald polynomials $P_\mu(x)$ of m variables x and $P_\nu(y)$ of n variables:

$$P_\lambda(x,y) = \sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n} b^\lambda_{\mu,\nu}\, P_\mu(x)P_\nu(y). \tag{7.3}$$

The expansion coefficients $b^\lambda_{\mu,\nu}$ are called the *branching coefficients*. Note that $b^\lambda_{\mu,\nu}=0$ unless $|\lambda|=|\mu|+|\nu|$. It is known that, when $n=1$, the branching coefficients are expressed by the Pieri coefficients $\psi_{\lambda/\mu}=\psi_{\lambda/\mu}(q,t)$:

$$P_\lambda(x,y) = \sum_{\mu\in\mathcal{P}_n,\,l\in\mathbb{N}} \psi_{\lambda/\mu}\, P_\mu(x)y^l, \tag{7.4}$$

where the sum is over the *horizontal strips* λ/μ with $\ell(\mu)\le m$.

NB: For each $\lambda \in \mathcal{P}_n$, there exists an irreducible polynomial representation of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ with highest weight λ, denoted by $V(\lambda)$, whose character is the Schur function $s_\lambda(x)$.

$$\mathrm{ch}_{V(\lambda)}(x) = \mathrm{tr}(g_x \cdot : V(\lambda) \to V(\lambda)) = s_\lambda(x), \tag{7.5}$$

where $g_x = \mathrm{diag}(x_1, \ldots, x_n)$ denotes a general element of the diagonal subgroup $\mathrm{T}_n \subset \mathrm{GL}_n$. In this context where $t = q$, the Littlewood-Richard coefficient $c^\lambda_{\mu,\nu}$ $(\lambda, \mu, \nu \in \mathcal{P}_n)$ are non-negative integers, and they represent the multiplicities of $V(\lambda)$ in the irreducible decomposition of the tensor product $V(\mu) \otimes V(\nu)$:

$$V(\mu) \otimes V(\nu) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V(\lambda)^{\oplus c^\lambda_{\mu,\nu}} \qquad (\text{as } \mathrm{GL}_n\text{-modules}). \tag{7.6}$$

On the other hand, for $\lambda \in \mathcal{P}_{m+n}$, $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$, the branching coefficients $b^\lambda_{\mu,\nu}$ are non-negative integers, and they represent the multiplicity of $V(\mu) \otimes V(\nu)$ in the restriction of $V(\lambda)$ from GL_{m+n} to the subgroup $\mathrm{GL}_m \times \mathrm{GL}_n$:

$$\mathrm{Res}\big\downarrow^{\mathrm{GL}_{m+n}}_{\mathrm{GL}_m \times \mathrm{GL}_n} (V(\lambda)) \simeq \bigoplus_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} \big(V(\mu) \otimes V(\nu)\big)^{\oplus b^\lambda_{\mu,\nu}} \qquad (\text{as } \mathrm{GL}_m \times \mathrm{GL}_n\text{-modules}). \tag{7.7}$$

7.2 Relation between $c^\lambda_{\mu,\nu}$ and $b^\lambda_{\mu,\nu}$.

Recall that the Macdonald polynomials have the kernel functions of Cauchy type, and of dual Cauchy type: For two sets of variables $z = (z_1, \ldots, z_M)$ and $w = (w_1, \ldots, w_N)$,

$$\begin{aligned} \Pi_{M,N}(z; w) &= \prod_{k=1}^{M} \prod_{l=1}^{N} \frac{(t z_k w_l; q)_\infty}{(z_k w_l; q)_\infty} = \sum_{\lambda \in \min\{M,N\}} b_\lambda P_\lambda(z) P_\lambda(w), \\ \Pi^\vee_{M,N}(z; w) &= \prod_{k=1}^{M} \prod_{l=1}^{N} (1 + z_k w_l) = \sum_{\lambda \subseteq (N^M)} P_\lambda(z|q,t) P_{\lambda'}(w|t,q). \end{aligned} \tag{7.8}$$

Theorem A Let $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$ and $\lambda \in \mathcal{P}_{m+n}$. Then we have $b_\lambda b^\lambda_{\mu,\nu} = b_\mu b_\nu c^\lambda_{\mu,\nu}$.

Theorem B Let $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$ and $\lambda \in \mathcal{P}_{m+n}$. Then we have $b^\lambda_{\mu,\nu}(q,t) = c^{\lambda'}_{\mu',\nu'}(t,q)$.

• **Proof of Theorem A:** Setting $M = N = m + n$ in (7.8), we have

$$\begin{aligned} \Pi_{m+n,N}(x, y; w) &= \sum_{\lambda \in \mathcal{P}_N} b_\lambda P_\lambda(x, y) P_\lambda(w) \\ &= \sum_{\lambda \in \mathcal{P}_N} \sum_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} b_\lambda b^\lambda_{\mu,\nu} P_\mu(x) P_\nu(y) P_\lambda(w) \end{aligned} \tag{7.9}$$

On the other hand,

$$\begin{aligned} \Pi_{m,N}(x; w) \Pi_{n,N}(y; w) &= \sum_{\mu \in \mathcal{P}_m} b_\mu P_\mu(x) P_\mu(w) \sum_{\nu \in \mathcal{P}_n} b_\nu P_\nu(y) P_\nu(w) \\ &= \sum_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} b_\mu b_\nu P_\mu(x) P_\nu(y) P_\mu(w) P_\nu(w) \\ &= \sum_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} b_\mu b_\nu P_\mu(x) P_\nu(y) \sum_{\lambda \in \mathcal{P}_N} c^\lambda_{\mu,\nu} P_\lambda(w) \\ &= \sum_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} \sum_{\lambda \in \mathcal{P}_N} b_\mu b_\nu c^\lambda_{\mu,\nu} P_\mu(x) P_\nu(y) P_\lambda(w). \end{aligned} \tag{7.10}$$

Since $\Pi_{m+n,N}(x,y;w)=\Pi_{m,N}(x;w)\Pi_{n,N}(y;w)$, we obtain $b_\lambda b^\lambda_{\mu,\nu}=b_\mu b_\nu c^\lambda_{\mu,\nu}$.

• **Proof of Theorem B:** Setting $M=m+n$ in (7.8), we have

$$\begin{aligned}\Pi^\vee_{m+n,N}(x,y;w)&=\sum_{\lambda\subseteq(N^{m+n})}P_\lambda(x,y)P^\circ_{\lambda'}(w)\\&=\sum_{\lambda'\in\mathcal{P}_N}\sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n}b^\lambda_{\mu,\nu}P_\mu(x)P_\nu(y)P^\circ_{\lambda'}(w)\end{aligned}\tag{7.11}$$

where $\circ$ denotes the operation of exchanging the parameters q and t. On the other hand

$$\begin{aligned}\Pi^\vee_{m,N}(x;w)\,\Pi^\vee_{n,N}(y;w)&=\sum_{\mu\subseteq(N^m)}P_\mu(x)P^\circ_{\mu'}(w)\sum_{\nu\subseteq(N^n)}P_\nu(y)P^\circ_{\nu'}(w)\\&=\sum_{\mu\subseteq(N^m),\,\nu\subseteq(N^n)}P_\mu(x)P_\nu(y)P^\circ_{\mu'}(w)P^\circ_{\nu'}(w)\\&=\sum_{\mu\subseteq(N^m),\,\nu\subseteq(N^n)}\sum_{\lambda'\in(N^{m+n})}\big(c^{\lambda'}_{\mu',\nu'}\big)^\circ P_\mu(x)P_\nu(y)P^\circ_{\lambda'}(w)\end{aligned}\tag{7.12}$$

Since $\Pi^\vee_{m+n,N}(x,y;w)=\Pi^\vee_{m,N}(x;w)\Pi^\vee_{n,N}(y;w)$, we obtain $b^\lambda_{\mu,\nu}=\big(c^{\lambda'}_{\mu',\nu'}\big)^\circ$.

• **Branching rule** (7.4)**:** For $l=0,1,\ldots$, we have

$$b^\lambda_{\mu,(l)}=\big(c^{\lambda'}_{\mu',(1^l)}\big)^\circ=(\psi'_{\lambda',\mu'})^\circ=\psi_{\lambda/\mu}\qquad(|\lambda|=|\mu|+l),\tag{7.13}$$

which proves the branching rule (7.4).

• **Comment of the evaluation of b_λ:** In view of the compatibility (6.9) of $P_\lambda(x)$ with respect to the number of variables, Macdonald [1] introduces *Macdonald functions* $P_\lambda(x)=P_\lambda(x|q,t)$ in infinite varialbes $x=(x_i)_{i\geq1}=(x_1,x_2,\ldots)$. Letting $M\to\infty$ and $N\to\infty$ in (7.8), we have

$$\begin{aligned}\Pi(z;w)&=\prod_{i\geq1}\prod_{j\geq1}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}=\sum_{\lambda\in\mathcal{P}}b_\lambda P_\lambda(x)P_\lambda(x),\\\Pi^\vee(x;y)&=\prod_{i\geq1}\prod_{j\geq1}(1+x_iy_j)=\sum_{\lambda\in\mathcal{P}}P_\lambda(x)P^\circ_{\lambda'}(y),\end{aligned}\tag{7.14}$$

where $\mathcal{P}$ denotes the set of all partitions $\lambda=(\lambda_1,\lambda_2,\ldots)$ ($\lambda_1\geq\lambda_2\geq\cdots;\ \lambda_i=0$ for $i\gg1$), and $P^\circ_\mu(x)=P_\mu(x|t,q)$. In terms of the power sums $p_k(x)=x_1^k+x_2^k+\cdots$, the kernel functions $\Pi(x;y)$ and $\Pi^\vee(x;y)$ are expressed as

$$\Pi(x;y)=\exp\Big(\sum_{k=1}^\infty\frac{1}{k}\frac{1-t^k}{1-q^k}p_k(x)p_k(y)\Big),\quad\Pi^\vee(x;y)=\exp\Big(\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k}p_k(x)p_k(y)\Big).\tag{7.15}$$

Denoting $\Lambda=\mathbb{C}[p_1,p_2,\ldots]$ the ring of symmetric functions in infinite variables, Macdonald introduces the *involution* $\omega_{q,t}:\ \Lambda\to\Lambda$ (algebra automorphism) by

$$\omega_{q,t}(p_k)=(-1)^{k-1}\frac{1-q^k}{1-t^k}p_k\qquad(k=1,2,\ldots)\tag{7.16}$$

in terms of the power sums, so that

$$\omega_{q,t}^{y}(\Pi(x;y)) = \Pi^{\vee}(x;y), \tag{7.17}$$

where $\omega_{q,t}^{y}$ denotes the involution $\omega_{q,t}$ acting on y variables. This implies

$$\sum_{\lambda\in\mathcal{P}} b_\lambda P_\lambda(x)\omega_{q,t}^{y}(P_\lambda(y)) = \sum_{\lambda\in\mathcal{P}} P_\lambda(x)P_{\lambda'}^{\circ}(y), \quad \text{namely} \quad b_\lambda\, \omega_{q,t}(P_\lambda) = P_{\lambda'}^{\circ}. \quad (\lambda\in\mathcal{P}) \tag{7.18}$$

In Macdonald [1], the explicit formula

$$b_\lambda = \prod_{s\in\lambda} \frac{1-t^{l_\lambda(s)+1}q^{a_\lambda(s)}}{1-t^{l_\lambda(s)}q^{a_\lambda(s)+1}} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{(t^{j-i+1}q^{\lambda_i-\lambda_j};q)_{\lambda_j-\lambda_{j+1}}}{(t^{j-i}q^{\lambda_i-\lambda_j+1};q)_{\lambda_j-\lambda_{j+1}}}. \tag{7.19}$$

for b_λ is proved by a somewhat tricky argument based on the compatibility of $b_\lambda\, \omega_{q,t}(P_\lambda) = P_{\lambda'}^{\circ}$ with the evaluation formula of $P_\lambda(1,t,\ldots,t^{n-1})$ in n-variables.

7.3 Tableau representation of $P_\lambda(x)$

We already know that the Macdonald polynomials $P_\lambda(x)$ $(\lambda\in\mathcal{P}_n)$ of n variables $x=(x_1,\ldots,x_n)$ is expressed as

$$P_\lambda(x_1,\ldots,x_{n-1},x_n) = \sum_{\substack{\mu\in\mathcal{P}_{n-1}\\ \mu\subseteq\lambda,\ \lambda/\mu:\,\text{h.s.}}} \psi_{\lambda/\mu}\ P_\mu(x_1,\ldots,x_{n-1})\, x_n^{|\lambda/\mu|} \tag{7.20}$$

whete the sum is taken over all partitions $\mu\in\mathcal{P}_{n-1}$ such that $\mu\subseteq\lambda$ and λ/μ is a horizontal strip. Note that $\psi_{\lambda/\mu}=0$ unless λ/μ is a horizontal strip. Repeating this procedure, one can express $P_\lambda(x)$ as a sum

$$P_\lambda(x) = \sum_{\phi=\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\cdots\subseteq\lambda^{(n)}=\lambda} \prod_{k=1}^{n} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}\ x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|} \tag{7.21}$$

over all nondecreasing sequences $\lambda^{(k)}$ $(k=0,1,\ldots,n)$ of partitions connecting ϕ and λ by n steps such that the skew partitions $\lambda^{(k)}/\lambda^{(k-1)}$ are horizontal strips. This representation can be interpreted as the sum

$$P_\lambda(x) = \sum_{T\in\mathrm{SSTab}_n(\lambda)} \psi_T\ x^{\mathrm{wt}(T)}, \quad \psi_T = \prod_{k=1}^{n} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}, \tag{7.22}$$

over all semi-standard tableau of shape λ. Here the coefficients ψ_T are expressed as

$$\psi_T = \prod_{1\le i\le j<k\le n} \frac{(t^{j-i+1}q^{\lambda_i^{(k-1)}-\lambda_j^{(k-1)}};q)_{\lambda_i^{(k)}-\lambda_i^{(k-1)}}}{(t^{j-i}q^{\lambda_i^{(k-1)}-\lambda_j^{(k-1)}+1};q)_{\lambda_i^{(k)}-\lambda_i^{(k-1)}}} \frac{(t^{j-i}q^{\lambda_i^{(k-1)}-\lambda_{j+1}^{(k)}+1};q)_{\lambda_i^{(k)}-\lambda_i^{(k-1)}}}{(t^{j-i+1}q^{\lambda_i^{(k-1)}-\lambda_{j+1}^{(k)}};q)_{\lambda_i^{(k)}-\lambda_i^{(k-1)}}}. \tag{7.23}$$

7.4 Macdonald-Ruijsenaars operators of row type

• **q-Difference operators $H_x^{(l)}$ of row type:** Let $\mathcal{R} = \mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}]$ be the commutative ring generated by the Macdonald-Ruijsenaars q-difference operators. Then, for each symmetric polynomial $f(\xi) \in \mathbb{C}[\xi]^{\mathfrak{S}_n}$, $\xi = (\xi_1, \ldots, \xi_n)$, there exists a unique q-difference operator $L_x \in \mathcal{R}$ such that

$$L_x P_\lambda(x) = f(t^\delta q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n). \tag{7.24}$$

(Express f as $f = F(e_1, \ldots, e_n)$ by a polynomial of $e_1, \ldots, e_n$. Then the operator L_x is given by $L_x = F(D_x^{(1)}, \ldots, D_x^{(n)})$.) This correspondence $L_x \to f$ defines an isomorphism $\mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}] \xrightarrow{\sim} \mathbb{C}[\xi]^{\mathfrak{S}_n}$ of commutative $\mathbb{C}$-algebras (a variation of the *Harish-Chandra isomorphism*).

For each $l = 0, 1, 2, \ldots$, we define a q-difference operator $H_x^{(l)}$ by

$$H_x^{(l)} = \sum_{\mu \in \mathbb{N}^n;\, |\mu| = l} \frac{\Delta(q^\mu x)}{\Delta(x)} \prod_{i,j=1}^{n} \frac{(t x_i/x_j; q)_{\mu_i}}{(q x_i/x_j; q)_{\mu_i}} T_{q,x}^{\mu}, \tag{7.25}$$

where $T_{q,x}^{\mu} = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$. Then one can show that $H_x^{(l)} \in \mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}]$, and that

$$H_x^{(l)} P_\lambda(x) = g_l(t^\delta q^\lambda) P_\lambda(x) \quad (l = 0, 1, 2 \ldots) \tag{7.26}$$

where $g_l(\xi)$ denotes the Macdonald polynomials attached to (l) of a single row.

$$g_l(\xi) = \sum_{\mu \in \mathbb{N}^n;\, |\mu| = l} \frac{(t;q)_{\mu_1} \cdots (t;q)_{\mu_n}}{(q;q)_{\mu_1} \cdots (q;q)_{\mu_n}} \xi_1^{\mu_1} \cdots \xi_n^{\mu_n} = \frac{(t;q)_l}{(q;q)_l} P_{(l)}(\xi) \quad (l = 0, 1, 2, \ldots). \tag{7.27}$$

In view of the generating function

$$\prod_{i=1}^{n} \frac{(t\xi_i u; q)_\infty}{(\xi_i u; q)_\infty} = \sum_{l=0}^{\infty} g_l(\xi) u^l \tag{7.28}$$

of Macdonald polynomials of single rows, we introduce the generating function $H_x(u) = \sum_{l=0}^{\infty} u^l H_x^{(l)}$. Then we have

$$H_x(u) P_\lambda(x) = P_\lambda(x) \prod_{i=1}^{n} \frac{(t^{n-i+1} q^\lambda u; q)_\infty}{(t^{n-i} q^\lambda u; q)_\infty} \tag{7.29}$$

• **Wronski relation:** Note that the generating functions

$$E(u) = \sum_{r=0}^{n} (-1)^r e_r(x) u^r = \prod_{i=1}^{n} (1 - x_i u), \quad G(u) = \sum_{l=0}^{\infty} g_l(x) u^l = \prod_{i=1}^{n} \frac{(t x_i u; q)_\infty}{(x_i u; q)_\infty} \tag{7.30}$$

satisfy $E(u)G(u) = E(tu)G(qu)$. This means that $e_r(x)$ and $g_l(x)$ are related through the recurrence relations

$$\sum_{r+l=k} (-1)^r (1 - t^r q^l) e_r(\xi) g_l(\xi) = 0 \quad (k = 1, 2, \ldots). \tag{7.31}$$

of *Wronski type*. One can verify that the operators $H_x^{(l)}$ $(l = 0, 1, 2, \ldots)$ defined above satisfy the Wronski relation

$$\sum_{r+l=k} (-1)^r (1 - t^r q^l) D_x^{(r)} H_x^{(l)} = 0 \quad (k = 1, 2, \ldots). \tag{7.32}$$

From this, it follows that $H_x^{(l)} \in \mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}]$ and that $H_x^{(l)}$ are diagonalized by the Macdonald polynomials as in (7.26). (See Noumi-Sano: arXiv:2012.03135)

• **Pieri formula of row type:** In the same way as we obtained the Pieri formula of column type from $D_x^{(r)}$, we can derive the Pieri formula of row type from

$$H_x^{(l)} = \sum_{|\mu|=l} H_\mu(x) T_{q,x}^{\mu}, \quad H_\mu(x) = \frac{\Delta(q^\mu x)}{\Delta(x)} \prod_{i,j=1}^{n} \frac{(tx_i/x_j;q)_{\mu_i}}{(qx_i/x_j;q)_{\mu_i}}. \tag{7.33}$$

In fact we have

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_\lambda(t^\delta q^{\mu+\nu}) = g_l(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu) \qquad (\lambda, \mu \in \mathcal{P}_n). \tag{7.34}$$

Since $H_\nu(t^\delta q^\mu) = 0$ unless $(\mu+\nu)/\mu$ is a horizontal strip, we obtain

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_{\mu+\nu}(t^\delta q^\lambda) = g_l(t^\delta q^\lambda) \widetilde{P}_\mu(t^\delta q^\lambda) \qquad (\lambda, \mu \in \mathcal{P}_n), \tag{7.35}$$

and hence

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_{\mu+\nu}(x) = g_l(x) \widetilde{P}_\mu(x) \qquad (\lambda, \mu \in \mathcal{P}_n), \tag{7.36}$$

This implies that

$$g_l(x) P_\mu(x) = \sum_{\lambda; |\lambda|=|\mu|+l} \varphi_{\lambda/\mu}\, P_\lambda(x), \qquad \varphi_{\lambda/\mu} = \frac{a_\mu}{a_\lambda} H_{\lambda-\mu}(t^\delta q^\mu). \tag{7.37}$$

Since $g_l(x) = b_{(l)} P_{(l)}(x)$, this means that

$$P_\mu(x) P_{(l)}(x) = \sum_{\lambda; |\lambda|=|\mu|+l} c_{\mu,(l)}^{\lambda}\, P_\lambda(x), \qquad c_{\mu,(l)}^{\lambda} = \frac{1}{b_{(l)}} \frac{a_\mu}{a_\lambda} H_{\lambda-\mu}(t^\delta q^\mu). \tag{7.38}$$

The corresponding branching coefficients are also detemined as

$$\psi_{\lambda/\mu} = b_{\mu,(l)}^{\lambda} = \frac{b_\mu b_{(l)}}{b_\lambda} c_{\mu,(l)}^{\lambda} = \frac{a_\mu b_\mu}{a_\lambda b_\lambda} H_{\lambda-\mu}(t^\delta q^\mu). \tag{7.39}$$