Introduction to Macdonald polynomials: Lecture 8

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8 Affine Hecke algebra and q-Dunkl operators

We denote by $W = \mathfrak{S}_n$ the symmetric group of degree n (*Weyl group* of type A_{n-1}), following the convention of Macdonald-Cherednik theory for general root systems. In this section, we denote by $\tau_i = T_{q,x_i}$ $(i = 1, \ldots, n)$ the q-shift operators in variables x_i, in order to avoid the conflict with the generators T_i of Hecke algebras. Setting $\tau = (\tau_1, \ldots, \tau_n)$, we denote by $\mathcal{D}_{q,x} = \mathbb{C}(x)[\tau^{\pm 1}]$ the algebra of q-difference operators in x variables with rational coefficients, and by $\mathcal{D}_{q,x}[W]$ the algebra of all operators of the form,

$$A_x = \sum_{\mu \in P,\, w \in W} a_{\mu,w}(x)\, \tau^{\mu}\, w \quad \text{(finite sum)}, \quad a_{\mu,w}(x) \in \mathbb{C}(x) \quad (\mu \in P,\ w \in W), \tag{8.1}$$

called the *q-difference-reflection operators*, where $P = \mathbb{Z}^n$ (*weight lattice* of GL_n), and for $\mu = (\mu_1, \ldots, \mu_n) \in P$, $\tau^{\mu} = \tau_1^{\mu_1} \cdots \tau_n^{\mu_n}$. Through their natural action on rational functions, we regard $\mathcal{D}_{q,x}[W]$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(x))$.

8.1 Affine Weyl groups and affine Hecke algebras

We denote by $\tau^P = \{\tau^{\mu} \mid \mu \in P\}$ the group of q-shift operators (*translations*) by P, and define the *extended affine Weyl group* $\widetilde{W}$ by

$$\widetilde{W} = \tau^P \rtimes W = \{\tau^{\mu}\, w \mid \mu \in P, w \in W\}\,; \qquad w\tau^{\mu} = \tau^{w.\mu} w \quad (\mu \in P,\ w \in W), \tag{8.2}$$

which is an extension of the standard *affine Weyl group* $W^{\mathrm{aff}} = \tau^{Q} \rtimes W$ with the group of translations by $Q = \{\mu \in P \mid |\mu| = \mu_1 + \cdots + \mu_n\}$ (*root lattice*). Denoting the canonical basis of P by ε_i $(i = 1, \ldots, n)$, we use the notation $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$ of the *simple roots*, so that $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{n-1} \subset P$. The idea of (affine) Hecke algebras is to construct t-deformations of the groups

$$W = \mathfrak{S}_n \subset W^{\mathrm{aff}} = \tau^{Q} \rtimes W \subset \widetilde{W} = \tau^{P} \rtimes W \subset \mathcal{D}_{q,x}[W] \tag{8.3}$$

within the algebra $\mathcal{D}_{q,x}[W]$ of q-difference-reflection operators. Note that the group-ring of $\widetilde{W}$

$$\mathbb{C}[\widetilde{W}] = \mathbb{C}[\tau^{P} \rtimes W] = \mathbb{C}[\tau^{\pm 1}][W] \subset \mathcal{D}_{q,x}[W] \tag{8.4}$$

is the ring of q-difference-reflection operators with constant coefficients.

We denote by

$$s_1 = (12),\ s_2 = (23),\ \ldots,\ s_{n-1} = (n-1, n) \tag{8.5}$$

the adjacent transpositions in $W = \mathfrak{S}_n$ (*simple reflections*) so that $W = \langle s_1, \ldots, s_{n-1} \rangle$. Note that s_i acts on the x variables by exchanging x_i and x_{i+1}. Besides these generators, we use the *affine reflection* s_0 and the *diagram automorphism* ω by setting

$$s_0 = \tau_1^{-1}\tau_n(1,n) \in W^{\mathrm{aff}}, \quad \omega = \tau_n(n-1,\ldots,1) = \tau_n s_{n-1}\cdots s_1 \in \widetilde{W} \tag{8.6}$$

where $(1,n)$ stands for the transposition of 1 and n, and $(n,n-1,\ldots,1)$ for the cyclic permutation $n \to n-1 \to \cdots \to 1 \to n$. These s_0 and ω are characterized as field automorphisms of $\mathbb{C}(x)$ acting on the x-variables by

$$\begin{aligned} &s_0(x_1) = qx_n, \quad s_0(x_i) = x_i \;\; (i = 2,\ldots,n-1), \quad s_0(x_n) = q^{-1}x_1 \\ &\omega(x_1) = qx_n, \quad \omega(x_2) = x_1, \quad \ldots, \quad \omega(x_n) = x_{n-1}. \end{aligned} \tag{8.7}$$

If fact, it is known that the three groups in (8.3) are generate by these operators as

$$W = \langle s_1,\ldots,s_{n-1}\rangle \subset W^{\mathrm{aff}} = \langle s_0, s_1,\ldots,s_{n-1}\rangle \subset \widetilde{W} = \langle s_0, s_1, \ldots, s_{n-1}, \omega\rangle, \tag{8.8}$$

with the fundamental relations:

$$\begin{aligned} &(0) \quad s_i^2 = 1 \quad (i = 0,1,\ldots,n-1) \\ &(1) \quad s_i s_j = s_j s_i \quad (j \not\equiv i, i\pm 1 \mod n) \\ &(2) \quad s_i s_j s_i = s_j s_i s_j \quad (j = i \pm 1 \mod n) \\ &(3) \quad \omega s_i = s_{i-1}\omega \quad (i = 1,\ldots,n-1), \qquad \omega s_0 = s_{n-1}\omega. \end{aligned} \tag{8.9}$$

In terms of these generators, the q-shift operators τ_i $(= T_{q,x_i})$ are expressed as follows:

$$\tau_1 = s_1\cdots s_{n-1}\omega, \quad \tau_2 = s_2\cdots s_{n-1}\omega s_1, \quad \cdots, \quad \tau_n = \omega s_1 \cdots s_{n-1}. \tag{8.10}$$

We now defines the q-difference-reflection operators T_i $(i = 0,1\ldots,n-1)$ by

$$T_i = t^{-\frac{1}{2}}\frac{1 - tx_i/x_{i+1}}{1 - x_i/x_{i+1}}(s_i - 1) + t^{\frac{1}{2}} = t^{-\frac{1}{2}}\frac{1 - tx_i/x_{i+1}}{1 - x_i/x_{i+1}}s_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - x_i/x_{i+1}} \tag{8.11}$$

for $i = 1,\ldots,n-1$ and

$$T_0 = t^{-\frac{1}{2}}\frac{1 - tqx_n/x_i}{1 - qx_n/x_1}(s_0 - 1) + t^{\frac{1}{2}} = t^{-\frac{1}{2}}\frac{1 - tqx_n/x_i}{1 - qx_n/x_1}s_0 + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - qx_n/x_1}. \tag{8.12}$$

Note that $x_i/x_{i+1} = x^{\alpha_i}$ $(i = 1,\ldots,n-1)$ correspond to the simple roots, and $qx_n/x_1 = x^{\alpha_0}$ to the *simple affine root* $\alpha_0 = \delta - \varepsilon_1 + \varepsilon_n$ with the convention $x^\delta = q$.

Theorem A: These operators T_i $(i = 0,1,\ldots,n-1)$ in $\mathcal{D}_{q,x}[W]$ together with ω satisfy the following relations.

$$\begin{aligned} &(0) \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 1 \quad (i = 0,1,\ldots,n-1) \\ &(1) \quad T_i T_j = T_j T_i \quad (j \not\equiv i, i\pm 1 \mod n) \\ &(2) \quad T_i T_j T_i = T_j T_i T_j \quad (j = i\pm 1 \mod n) \\ &(3) \quad \omega T_i = T_{i-1}\omega \quad (i = 1,\ldots,n-1). \qquad \omega T_0 = T_{n-1}\omega. \end{aligned} \tag{8.13}$$

In this way, we obtain t-deformations of the group-rings of W, W^{aff}, $\widetilde{W}$ in $\mathcal{D}_{q,x}[W]$ as follows:

$$\begin{array}{ccc} \mathbb{C}[W] = \mathbb{C}\langle s_1, \ldots, s_{n-1}\rangle & & H[W] = \mathbb{C}\langle T_1, \ldots, T_{n-1}\rangle \\ \cap & & \cap \\ \mathbb{C}[W^{\mathrm{aff}}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}\rangle & & H[W^{\mathrm{aff}}] = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}\rangle \\ \cap & & \cap \\ \mathbb{C}[\widetilde{W}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}, \omega^{\pm 1}\rangle & & H[\widetilde{W}] = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}, \omega^{\pm 1}\rangle \end{array} \tag{8.14}$$

Here $H(W)$ denotes the Hecke algebra associate with the Weyl group W; $H(W^{\mathrm{aff}})$ and $H(\widetilde{W})$ are called (extended) *affine Hecke algebras.* Note that the fundamental relations $s_i^2 = 1$ are replaced by the quadratic relations $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$, and hence $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.

8.2 q-Dunkl operators

In view of the two expressions

$$\mathbb{C}[\widetilde{W}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}, \omega^{\pm 1}\rangle = \mathbb{C}[\tau^{\pm 1}][W] \tag{8.15}$$

of the group-ring of $\widetilde{W}$, it would be natural to ask what are the "translations" in $H(\widetilde{W})$. Imitating the formulas (8.10) for τ_i $(i = 1, \ldots, n)$, we define the *q-Dunkl operators* (or *Cherednik operators*) $Y_1, \ldots, Y_n$ by

$$Y_1 = T_1 \cdots T_{n-1}\omega, \quad Y_2 = T_2 \cdots T_{n-1}\omega T_1^{-1}, \quad \ldots, \quad Y_n = \omega T_1^{-1} \cdots T_{n-1}^{-1} \in H(\widetilde{W}). \tag{8.16}$$

Notice here that T_i are replaced by their inverses T_i^{-1} when they are located to the right side of ω.

Theorem B: The q-Dunkl operators $Y_1, \ldots, Y_n \in H(\widetilde{W})$ commute with each other. Furthermore, they generate a commutative subalgebra $\mathbb{C}[Y^{\pm 1}] = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \subset H(\widetilde{W})$ isomorphic to the algebra of Laurent polynomials in n variables.

One can directly verify the commutativity $Y_iY_j = Y_jY_i$ $(i, j \in \{1, \ldots, n\})$ by the definition (8.16) and the fundamental relations of T_i, ω in (8.13). With this "translation subalgebra" of q-Dunkl operators, the extended affine Heck algebra is expressed in the form

$$H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}] \otimes H(W) = \bigoplus_{w \in W} \mathbb{C}[Y^{\pm 1}]\, T_w, \tag{8.17}$$

where, for each $w \in W$, T_w is the element defined as $T_w = T_{s_{i_1}} \cdots T_{s_{i_l}}$ in terms of a reduced (shortest) expression $w = s_{i_1} \cdots s_{i_l}$ of w; this definition does not depend on the choice of the reduced expression (Lemma of Iwahori-Matsumoto). From this, we see that one can take $T_1, \ldots, T_{n-1}$ and the q-Dunkl operators $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ for the generators of the extended affine Heck algebra:

$$H(\widetilde{W}) = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}; \omega^{\pm 1}\rangle = \mathbb{C}\langle T_1, \ldots, T_n; Y_1^{\pm 1}, \ldots Y_n^{\pm 1}\rangle. \tag{8.18}$$

Theorem C (Bernstein): The center $\mathcal{Z}H(\widetilde{W})$ of the extended affine Hecke algebra is precisely the W-invariant part of the commutative algebra of q-Dunkl operators:

$$\mathcal{Z}H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}]^W = \left\{ f(Y) \mid f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W \right\}, \quad \xi = (\xi_1, \ldots, \xi_n). \tag{8.19}$$

8.3 From q-Dunkl operators to Macdonald-Ruijsenaars operators

Let $A_x \in \mathcal{D}_{q,x}[W]$ be a q-difference-reflection operator in the form

$$A_x = \sum_{\mu\in P,\, w\in W} a_{\mu,w}(x)\,\tau^\mu\, w \quad \text{(finite sum)}, \quad a_{\mu,w}(x) \in \mathbb{C}(x) \quad (\mu \in P,\ w \in W). \tag{8.20}$$

If $\varphi(x)$ is a symmetric (W-invariant) function, A_x acts on $\varphi(x)$ as a q-difference operator: since $w\varphi(x) = \varphi(x)$ ($w \in W$), we have

$$A_x\varphi(x) = \sum_{\mu\in P,\, w\in W} a_{\mu,w}(x)\,\tau^\mu \varphi(x) = L_x\varphi(x), \qquad L_x = \sum_{\mu\in P,\, w\in W} a_{\mu,w}(x)\,\tau^\mu. \tag{8.21}$$

In order to describe the action of q-Dunkl operators, we set

$$c(z) = t^{-\frac{1}{2}}\frac{1-tz}{1-z}, \quad d_\pm(z) = t^{\pm\frac{1}{2}} - c(z) \tag{8.22}$$

so that

$$T_i^{\pm 1} = c(x_i/x_{i+1})s_i + d_\pm(x_i/x_{i+1}) \quad (i = 1,\ldots,n-1), \quad T_0^{\pm 1} = c(qx_n/x_q)s_i + d_\pm(qx_n/x_1). \tag{8.23}$$

Note also that $c(z)$ satisfies $c(z) + c(z^{-1}) = t^{\frac{1}{2}} + t^{-\frac{1}{2}}$.

• **Example ($\boldsymbol{n = 2}$):** In this case, we have two q-Dunkl operators.

$$\begin{aligned} Y_1 &= T_1\omega = c(x_1/x_2)s_1\omega + d_+(x_1/x_2)\omega = c(x_1/x_2)\tau_1 + d_+(x_1/x_2)\tau_2 s_1 \\ Y_2 &= \omega T_1^{-1} = \omega c(x_1/x_2)s_1 + \omega d_-(x_1/x_2) = c(qx_2/x_1)\tau_2 + d_-(qx_2/x_1)\tau_2 s_1 \end{aligned} \tag{8.24}$$

Then,

$$\begin{aligned} Y_1 + Y_2 &= c(x_1/x_2)\tau_1 + c(qx_2/x_1)\tau_2 + (d_-(qx_2/x_1) + d_+d_+(x_1/x_2))\tau_2 s_1 \\ Y_2Y_1 &= \omega^2 = \tau_1\tau_2 \end{aligned} \tag{8.25}$$

Since

$$c(qx_2/x_1) + d_-(qx_2/x_2) + d_+(x_1/x_2) = t^{-\frac{1}{2}} + t^{-\frac{1}{2}} - c(x_1/x_2) = c(x_2/x_1), \tag{8.26}$$

for any symmetric function $\varphi(x) = \varphi(x_1, x_2)$, we have

$$\begin{aligned} (Y_1 + Y_2)\varphi(x) &= (c(x_1/x_2)\tau_1 + c(x_2/x_1)\tau_2)\varphi(x) \\ &= t^{-\frac{1}{2}}\Big(\frac{1-tx_1/x_2}{1-x_1/x_2}\tau_1 + \frac{1-tx_2/x_1}{1-x_2/x_1}\tau_2\Big)\varphi(x) \\ &= t^{-\frac{1}{2}}D_x^{(1)}\varphi(x) \\ (Y_2Y_1)\varphi(x) &= \tau_1\tau_2\varphi(x) = t^{-1}D_x^{(2)}\varphi(x), \end{aligned} \tag{8.27}$$

where $D_x^{(1)}$, $D_x^{(2)}$ are the Macdonald-Ruijsenaars operators in two variables.

For each $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]$, there exists a unique q-difference operator $L_x^f \in \mathcal{D}_{q,x}$ such that

$$f(Y)\varphi(x) = L_x^f\varphi(x) \tag{8.28}$$

for any symmetric function $\varphi(x)$; express $A_x = f(Y)$ in the form (8.20), and take $L_x = L_x^f$ as in (8.21).

Theorem D: For any $f \in \mathbb{C}[\xi^{\pm 1}]^W$, $L_x^f \in \mathcal{D}_{q,x}$ is a W-invariant q-difference operator. Furthermore, L_x^f $(f \in \mathbb{C}[\xi^{\pm 1}]^W)$ commute with each other: $L_x^f L_x^g = L_x^g L_x^f$ for any $f, g \in \mathbb{C}[\xi^{\pm 1}]^W$.

Let's take the elementary symmetric functions $e_r(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W$ $(r = 1, \ldots, n)$ for $f(\xi)$. Then, one can show that the q-Dunkl operators

$$e_r(Y) = \sum_{1 \le i_1 < \cdots < i_r \le n} Y_{i_1} \cdots Y_{i_r} \tag{8.29}$$

induce a commuting family of W-invariant q-difference operator $L_x^{e_r}$ of the form

$$L_x^{e_r} = \Big(\prod_{\substack{1 \le i \le r \\ r+1 \le j \le n}} t^{-\frac{1}{2}} \frac{1 - t x_i / x_j}{1 - x_i / x_j} \Big) \tau_1 \cdots \tau_r + \cdots . \tag{8.30}$$

This implies that $L_x^{e_r}$ are constant multiples of $D_x^{(r)}$ respectively,

$$L_x^{e_r} = t^{-\frac{1}{2}(n-1)r} D_x^{(r)} \quad (r = 0, 1, \ldots, n), \tag{8.31}$$

and also that they are diagonalized by the Macdonald polynomials:

$$L_x^{e_r} P_\lambda(x) = e_r(t^\rho q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n), \tag{8.32}$$

where $\rho = \frac{1}{2} \sum_{i=1} (n - 2i + 1) \varepsilon_i = \delta - \frac{1}{2}(n-1)(1^n)$.

To summarize: There is an isomorphism of algebras

$$\mathcal{Z}H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}]^W \stackrel{\sim}{\to} \mathbb{C}[D_1^{(1)}, \ldots, D_x^{(n)}, (D_x^{(n)})^{-1}] : \quad f(Y) \to L_x^f \tag{8.33}$$

from the algebra of symmetric q-Dunkl operators to the algebra of Macdonald-Ruijsenaars operators. Furthermore, for all $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W$, we have

$$f(Y) P_\lambda(x) = L_x^f P_\lambda(x) = f(t^\rho q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n). \tag{8.34}$$

One can directly check that the operators T_i $(i = 0.1, \ldots, n-1$ stabilize the algebra $\mathbb{C}[x^{\pm 1}]$ of Laurent polynomials in $x = (x_1, \ldots, x_n)$. Hence $\mathbb{C}[x^{\pm 1}]$ can be regarded as a left $H(\widetilde{W})$-module. It would be natural to anticipate that the commutative subalgebra $\mathbb{C}[Y^{\pm 1}]$ of $H(\widetilde{W})$ can be simultaneously diagonalized on $\mathbb{C}[x^{\pm 1}]$. In fact, the q-Dunkl operators have common eigenfunctions $E_\mu(x)$ $(\mu \in P)$, called the *nonsymmetric Macdonald polynomials*.

In the following, we denote by

$$P_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in P \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n\} = \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_{n-1} \oplus \mathbb{Z}\varpi_n, \tag{8.35}$$

the cone of *dominant integral weights*, where, for $r = 1, \ldots, n$, $\varpi_r = (1^r) = \varepsilon_1 + \cdots + \varepsilon_r$ (*fundamental weights*). Then, for each $\mu \in P$, there exists a unique $\mu_+ \in P_+$ in the W-orbit of μ: $W.\mu \cap P_+ = \{\mu_+\}$. For the diagonalization of the q-Dunkl operators, we make use of the partial order

$$\mu \preceq \nu \quad \Longleftrightarrow \quad \mu_+ < \nu_+ \ \text{ or } \ (\mu_+ = \nu_+ \text{ and } \ \mu \le \nu). \tag{8.36}$$

defined by applying the dominance order in two steps.

Theorem E: Assume that $t \in \mathbb{C}^*$ is generic. Then there exists a unique Laurent polynomial $E_\mu(x) \in \mathbb{C}[x^{\pm 1}]$ such that

(1) $f(Y)E_\mu(x) = f(t^{\rho_\mu} q^\mu)E_\mu(x)$ for all $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]$,

(2) $E_\mu(x) = x^\mu +$ (lower order terms with respect to $\preceq$).

Then, regarded as a $H(\widetilde{W})$-module, the algebra of Laurent polynomials $\mathbb{C}[x^{\pm 1}]$ is decomposed into irreducible components as follows:

$$\mathbb{C}[x^{\pm 1}] = \bigoplus_{\lambda \in P_+} V(\lambda), \quad V(\lambda) = \bigoplus_{\mu \in W.\lambda} \mathbb{C}E_\mu(x). \tag{8.37}$$

Furthermore, for each $\lambda \in P_+$, we have

$$V(\lambda)^{H(W)} = \left\{ v \in V(\lambda) \mid T_i v = t^{\frac{1}{2}} v \quad (i = 1, \dots, n-1) \right\} = \mathbb{C}P_\lambda(x), \tag{8.38}$$

where $P_\lambda(x)$ is the Macdonald (Laurent) polynomial attached λ; if we take $l \in \mathbb{Z}$ and $\mu \in \mathcal{P}_n$ such that $\lambda = \mu + (l^n)$, then $P_\lambda(x)$ is expressed as $P_\lambda(x) = (x_1 \dots x_n)^l P_\mu(x)$ in terms of the Macdonald polynomial $P_\mu(x)$ attached to a partition $\mu \in \mathcal{P}_n$.

8.4 Double affine Hecke algebra and Cherednik involution

The algebra $\mathcal{D}_{q,x}[W]$ contains the subalgebra

$$\mathbb{C}[x^{\pm 1}; \tau^{\pm}][W] = \mathbb{C}\langle x_1^{\pm 1}, \dots, x^{\pm n}; s_1, \dots, s_{n-1}; \tau_1^{\pm 1}, \dots, \tau_n^{\pm 1} \rangle \subset \mathcal{D}_{q,x}[W] \tag{8.39}$$

of q-difference-reflection operators with Laurent polynomial coefficients. This algebra can be thought of as a q-version of $\mathbb{C}[x; \partial_x][W]$ (crossed product of the Heisenberg algebra and the Weyl group.) One can consider the t-deformation of this algebra

$$\begin{aligned} DH(\widetilde{W}) &= \mathbb{C}[x^{\pm 1}] \otimes H(W) \otimes \mathbb{C}[Y^{\pm 1}] \\ &= \mathbb{C}\langle x_1^{\pm 1}, \dots, x_n^{\pm 1}; T_1, \dots, T_{n-1}; Y_1^{\pm 1}, \dots, Y_n^{n-1} \rangle \subset \mathcal{D}_{q,x}[W], \end{aligned} \tag{8.40}$$

called the *double affine Hecke algebra*. This algebra consists of all operators of the form

$$A = \sum_{\mu, \nu \in P;\, w \in W} a_{\mu,w,\nu}\, x^\mu\, T_w\, Y^\nu \quad \text{(finite sum)} \quad (a_{\mu,w,\nu} \in \mathbb{C}). \tag{8.41}$$

Also, commutation relations between T_i and x_j, Y_j are given by

$$T_i x_i T_i = x_{i+1} \quad (i = 1, \dots, n-1), \quad T_i x_j = x_j T_i \quad (j \neq i, i+1). \tag{8.42}$$

and

$$T_i Y_{i+1} T_i = Y_i \quad (i = 1, \dots, n-1), \quad T_i Y_j = Y_j T_i \quad (j \neq i, i+1). \tag{8.43}$$

Theorem F (Cherednik): There exists a unique involutive anti-homomorphism $\phi: \ DH(\widetilde{W}) \to DH(\widetilde{W})$ such that

$$\phi(x_i) = Y^{-i},\ \phi(Y_i) = x_i^{-1}\ (i = 1, \dots, n), \qquad \phi(T_i) = T_i \quad (i = 1, \dots, n-1). \tag{8.44}$$

This anti-involution ϕ is called the *Cherednik involution* ($\phi(ab)=\phi(b)\phi(a)$, $\phi^2=1$).

We define the *expectation value* $\langle\ \rangle : DH(\widetilde{W}) \to \mathbb{C}$ by

$$\langle A\rangle = A(1)\big|_{x=t^{-\rho}} = \sum_{\mu,\nu\in P; w\in W} a_{\mu,w,\nu} t^{-\langle\rho,\mu\rangle} t^{\frac{1}{2}\ell(w)} t^{\langle\rho,\nu\rangle}, \tag{8.45}$$

where $\ell(w)$ the length of w (= the number of pairs (i,j) $(1\le i<j\le n)$ such that $w(i)>w(j)$). We also define a scalar product (bilinear form)

$$\langle\ ,\ \rangle :\quad DH(\widetilde{W}) \times DH(\widetilde{W}) \to \mathbb{C} \tag{8.46}$$

by

$$\langle A,B\rangle = \langle\phi(A)B\rangle \in \mathbb{C} \qquad (A,B\in DH(\widetilde{W})). \tag{8.47}$$

By the definition of the Cherednik involution, we have

$$\langle\phi(A)\rangle = \langle A\rangle, \quad \langle A,B\rangle = \langle B,A\rangle. \tag{8.48}$$

(This bilinear form is a variation of Fisher's scalar product.)

We apply formula (8.48) to Macdonald polynomials $A=P_\lambda(x)$ and $B=P_\mu(x)$ $\quad(\lambda,\nu\in P_+)$.

$$\begin{aligned}\langle P_\lambda(x),P_\mu(x)\rangle &= \langle\phi(P_\lambda(x))P_\mu(x)\rangle = \langle P_\lambda(Y^{-1})P_\mu(x)\rangle\\ &= \langle P_\lambda(t^{-\rho}q^{-\mu})P_\mu(x)\rangle = P_\lambda(t^{-\rho}q^{-\mu})P_\mu(t^{-\rho})\end{aligned} \tag{8.49}$$

Since $\langle P_\lambda(x),P_\mu(x)\rangle = \langle P_\mu(x),P_\lambda(x)\rangle$, we have

$$P_\lambda(t^{-\rho}q^{-\mu})P_\mu(t^{-\rho}) = P_\mu(t^{-\rho}q^{-\lambda})P_\lambda(t^{-\rho}) \qquad (\lambda,\mu\in P_+), \tag{8.50}$$

and hence

$$\frac{P_\lambda(t^{-\rho}q^{-\mu})}{P_\lambda(t^{-\rho})} = \frac{P_\mu(t^{-\rho}q^{-\lambda})}{P_\mu(t^{-\rho})}. \tag{8.51}$$

By the property $P_\lambda(x;q,t)=P_\lambda(x;q^{-1},t^{-1})$ of Macdonald polynomials, from (8.51) we obtain

$$\frac{P_\lambda(t^{\rho}q^{\mu})}{P_\lambda(t^{\rho})} = \frac{P_\mu(t^{\rho}q^{\lambda})}{P_\mu(t^{\rho})} \qquad (\lambda,\mu\in P_+). \tag{8.52}$$

Since $\rho=\delta-\frac{1}{2}(n-1)(1^n)$, this formula is identical to the self-duality we discussed in Section 5.