Lecture 8 : 2021/03/26

§7 : Pieri coefficients and branching coefficients

(III) Littlewood-Richardson coefficients

$x = (x_1, \dots, x_n)$

$$P_\mu(x) P_\nu(x) = \sum_{\lambda \in \mathcal{P}_n} c^\lambda_{\mu\nu} P_\lambda(x) \qquad (\mu, \nu \in \mathcal{P}_n)$$

$\nu = (1^r)$ $(r = 0, 1, \dots, n)$

$$P_\mu(x) e_r(x) = \sum_{\substack{\lambda/\mu : \text{v.s} \\ |\lambda/\mu| = r}} \psi'_{\lambda/\mu} \cdot P_\lambda(x)$$

Pieri formula

$c^\lambda_{\mu(1^r)}$
$= \psi'_{\lambda/\mu}$
$= \psi_{\lambda'/\mu'}(t, q)$

(11) Branching coefficients

$x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$

$$P_\lambda(x, y) = \sum_{\mu \in \mathcal{P}_m, \nu \in \mathcal{P}_n} b^\lambda_{\mu\nu} P_\mu(x) P_\nu(y)$$

$(m+n)$ variables

As we will see later:

$$P_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_{\mu \in \mathcal{P}} \psi_{\lambda/\mu} P_\mu(x_1, \dots, x_{n-1}) x_n^{|\lambda/\mu|}$$

$$b^\lambda_{\mu,(\ell)} = \psi_{\lambda/\mu} \quad |\lambda/\mu| = \ell$$

$$= \psi_{\lambda/\mu}(q, t)$$

Theorem A: $b_\lambda b^\lambda_{\mu\nu} = b_\mu b_\nu c^\lambda_{\mu\nu}$

$q \leftrightarrow t$

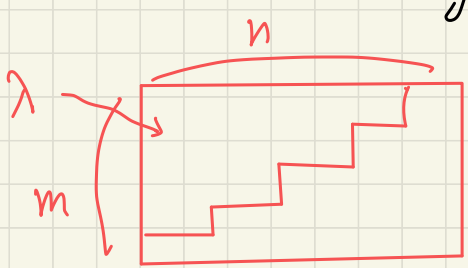
Theorem B: $b^\lambda_{\mu,\nu}(q,t) = c^{\lambda'}_{\mu',\nu'}(t,q)$ $\qquad b^\lambda_{\mu\nu} = (c^{\lambda'}_{\mu'\nu'})^\circ$

$x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$

$$\Pi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_{\ell(\lambda)\le \min\{m,n\}} b_\lambda P_\lambda(x) P_\lambda(y) \Rightarrow (A)$$

$$\Pi^\vee_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j) = \sum_{\lambda \subseteq (n^m)} P_\lambda(x) P^\circ_\lambda(y) \Rightarrow (B)$$

$\leftarrow m \times n$ rectangle



$$\text{②}\quad \Pi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_{\ell(\lambda)\le \min\{m,n\}} b_\lambda P_\lambda(x)P_\lambda(y) \Rightarrow (A)$$

$$b_\lambda c^\lambda_{\mu\nu} = b_\mu b_\nu c^\lambda_{\mu\nu}$$

$$x=(x_1,\dots,x_m),\ y=(y_1,\dots,y_n),\quad w=(w_1,\dots,w_{m+n})$$

$$\Pi_{m+n,m+n}(x,y;w) = \Pi_{m,m+n}(x;w)\,\Pi_{n,m+n}(y;w)$$

$$\Pi_{m+n,m+n}(x,y;w) = \sum_\lambda b_\lambda \boxed{P_\lambda(x,y)} P_\lambda(w) = \sum_{\lambda,\mu,\nu} b_\lambda \boxed{c^\lambda_{\mu\nu} P_\mu(x)P_\nu(y)} P_\lambda(w)$$

$$\Pi_{m,m+n}(x;w)\,\Pi_{n,m+n}(y;w) = \sum_\mu b_\mu P_\mu(x)P_\mu(w)\sum_\nu b_\nu P_\nu(y)P_\nu(w) = \sum_{\mu,\nu} b_\mu b_\nu P_\mu(x)P_\nu(y)\boxed{P_\mu(w)P_\nu(w)} = \sum_{\lambda,\mu,\nu} b_\mu b_\nu \boxed{c^\lambda_{\mu\nu}} P_\mu(x)P_\nu(y)\boxed{P_\lambda(w)}$$

(11) Tableau representation

$$b^{\lambda}_{\mu(\ell)} = \left(c^{\lambda'}_{\mu'(1^\ell)}\right)^{\circ} = \left(\psi'_{\lambda'/\mu'}\right)^{\circ} = \psi_{\lambda/\mu}.$$

Branching rule:

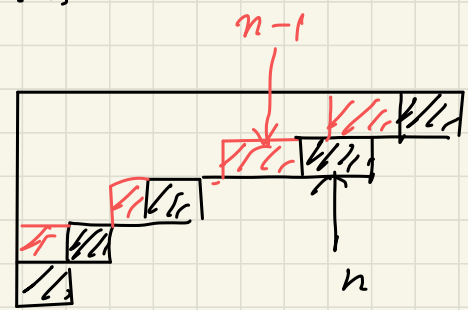
$$P_\lambda(x_1,\dots,x_{n-1},x_n) = \sum_{\substack{\mu \subseteq \lambda \\ h.s.}} \psi_{\lambda/\mu} P_\mu(x_1,\dots,x_{n-1})\, x_n^{|\lambda/\mu| = \ell}$$

$$= \sum_{\substack{\nu \subseteq \mu \subseteq \lambda \\ h.s \quad h.s}} \psi_{\lambda/\mu}\, \psi_{\mu/\nu}\, P_\nu(x_1,\dots,x_{n-2})\, x_{n-1}^{|\mu/\nu|}\, x_n^{|\lambda/\mu|}$$

$$= \cdots$$

$$= \sum_{\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda} \boxed{\prod_{k=1}^{n} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}}\, x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|}$$

$$= \sum_{T \in SSTab_n(\lambda)} \psi_T\, x^{wt(T)}$$



Remark Other combinatorial formulas

- Ram-Yip formula $\Leftarrow$ Double affine Hecke algebra
- Haglund formula ← Garsia, Haiman, Haglund...

modified Macdonald polynomials $x=(x_1,x_2,\dots)=(x_i)_{i\geq 1}$

$$\tilde{H}_\lambda(x|q,t) = \text{const.}\, P_\lambda\left(\text{"}\frac{x}{1-t}\text{"}\,|\,q,t\right)$$

← combinatorial formula / tableaux

$\text{"}\frac{x}{1-t}\text{"}$: replace $(x_i)_{i\geq 1}$ by $(t^r x_i)_{i\geq 1, r\geq 0}$

$$\frac{x_i}{1-t} = x_i + t x_i + t^2 x_i + \cdots = \sum_{r\geq 0} t^r x_i$$

$$p_\ell(x) = \sum_{i\geq 1} x_i^\ell$$

$$p_\ell\left(\text{"}\frac{x}{1-t}\text{"}\right) = \sum_{i\geq 1,\, r\geq 0} (t^r x_i)^\ell$$

$$= \frac{1}{1-t^\ell} \sum_{i\geq 1} x_i^\ell = \frac{1}{1-t^\ell} p_\ell(x)$$

§8: Affine Hecke algebras and q-Dunkl operators

permutation of variables

(11) Algebra of q-difference-reflection operators

$$\mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}] \qquad W = \mathfrak{S}_n \ (\text{Weyl group of type } A_{n-1})$$

$$\cap\!\!\mid$$

$$\mathcal{D}_{q,x}[W] = \left\{ A_x = \sum_{\substack{\mu \in P \\ w \in W}} a_{\mu}(x) \boxed{T_{q,x}^{\mu}} w \ (\text{finite sum}) \right\}$$

$\mathbb{C}(x)$ $\quad \tau^{\mu}$

w

q-shift operators

<u>Notation</u> $\tau_i = T_{q,x_i}$ $\quad (i = 1, \dots, n)$

$$P = \mathbb{Z}^n = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$$

$$\mu = (\mu_1, \dots, \mu_n); \quad \tau^{\mu} = \tau_1^{\mu_1} \cdots \tau_n^{\mu_n} \quad |\mu| = \mu_1 + \cdots + \mu_n$$

⊛ Affine Weyl groups and affine Hecke algebra

$$W = \mathfrak{S}_n \subset W^{aff} = \tau^Q \rtimes W \subset \widetilde{W} = \tau^P \rtimes W$$

semi-direct

$$w\tau^\mu = \tau^{w\mu} \cdot w$$

Weyl group affine Weyl group extended affine Weyl group

$\tau^P = \{\tau^\mu \mid \mu \in P\}$ translation group (q-shift op's)

free abelian group of rank n

$\tau^Q = \{\tau^\mu \mid \mu \in Q\}$ ← rank $n-1$

$Q = \{\mu \in P \mid |\mu| = \mu_1 + \cdots + \mu_n = 0\} \subset P$ ← weight lattice

root lattice

$= \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{n-1}$

$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$

<u>simple roots</u>

② Description of $W^{aff}, \widetilde{W}$ in terms of generators and relations

$s_1 = (12),\ s_2 = (23),\ \dots,\ s_{n-1} = (n-1, n)$ adjacent transpositions (simple reflections)

$$W = \mathfrak{S}_n = \langle s_1, s_2, \dots, s_{n-1} \rangle$$

$$\cap$$

$$W^{aff} = \langle s_0, s_1, \dots, s_{n-1} \rangle$$

$$\cap$$

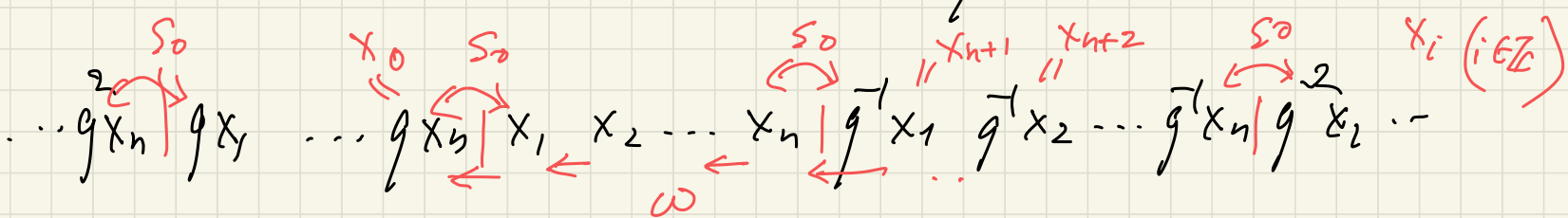
$$\widetilde{W} = \langle s_0, s_1, \dots, s_{n-1}, \omega \rangle$$

- $s_i : x_i \longleftrightarrow x_{i+1}$
- s_0: affine reflection

  $s_0(x_1) = q x_n \quad s_0(x_n) = q^{-1} x_1$

  $s_0(x_i) = x_i \ (i = 2, \dots, n)$
- ω: diagram rotation

  $\omega(x_1) = q x_n,\ \omega(x_i) = x_{i-1} \ (i = 2, \dots, n)$



$$W = \langle s_1, \dots, s_{n-1} \rangle \subset W^{\mathrm{aff}} = \langle s_0, s_1, \dots s_{n-1} \rangle \subset \widetilde{W} = \langle s_0, s_1, \dots s_{n-1}, \omega \rangle$$

$$= \tau^Q \rtimes W \qquad \mathbb{Z}/n\mathbb{Z} \qquad = \tau^P \rtimes W$$

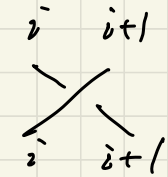
$\tau_1, \dots, \tau_n$

(0) $s_i^2 = 1$ (1) $s_i s_j = s_j s_i$ ($j \not\equiv i, i\pm 1 \bmod n$)

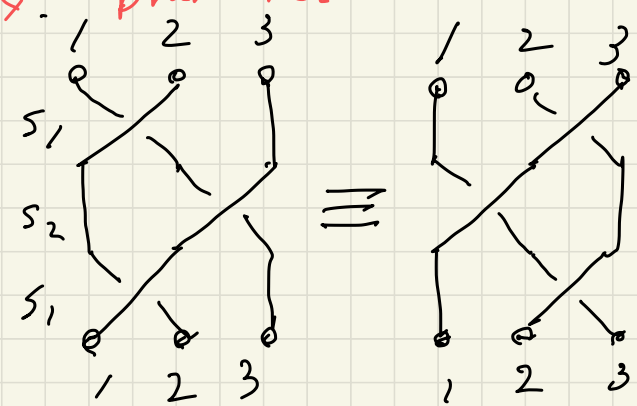
(2) $s_i s_j s_i = s_j s_i s_j$ ($j \equiv i \pm 1 \bmod n$)

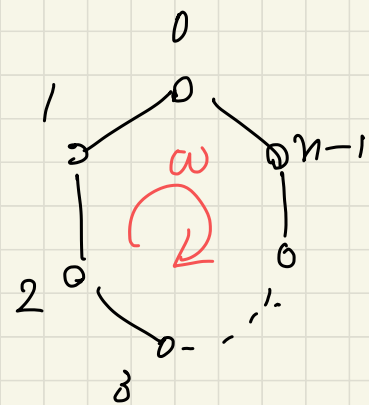
(3) $\omega s_0 = s_{n-1} \omega$, $\omega s_i = s_{i-1} \omega$ ($i = 1, \dots, n-1$)

s_i



braid relation





$x_2 \leftarrow x_1 \leftarrow \cdots \leftarrow x_1 \leftarrow x_2$

$$\begin{cases} \tau_1 = s_1 \cdots s_{n-1} \omega \\ qx_1 \leftarrow \cdots \leftarrow qx_{n-1} \leftarrow qx_n \leftarrow x_1 \\ \tau_2 = s_2 \cdots s_{n-1} \omega s_1 \\ \vdots \\ \tau_n = \omega s_1 \cdots s_{n-1} \end{cases}$$

(11) t-deformation of W^{aff}, $\tilde{W}$ $\Rightarrow$ affine Hecke algebras

$$\begin{array}{ccc}
\mathbb{C}[W] = \mathbb{C}\langle s_1, \dots, s_{n-1} \rangle & & H(W) = \mathbb{C}\langle T_1, \dots, T_{n-1} \rangle \\
\cap & & \cap \\
\mathbb{C}[W^{aff}] = \mathbb{C}\langle s_0, s_1, \dots, s_{n-1} \rangle & & H(W^{aff}) = \mathbb{C}\langle T_0, T_1, \dots, T_{n-1} \rangle \\
\cap & \Longrightarrow & \cap \\
\mathbb{C}[\tilde{W}] = \mathbb{C}\langle s_0, s_1, \dots, s_{n-1}, \omega^{\pm 1} \rangle & & H(\tilde{W}) = \mathbb{C}\langle T_0, T_1, \dots, T_{n-1}, \omega^{\pm 1} \rangle \\
\cap & & \cap \\
\mathcal{D}_{q,\chi}[W] & & \mathcal{D}_{q,\chi}[W]
\end{array}$$

$$\mathbb{C}[\tau^{\pm 1}][W] = \mathbb{C}[\tilde{W}]$$

① Definition of T_i (Demazure-Lusztig operators)

$t=1$ s_i

$$T_i = t^{-\frac{1}{2}} \frac{1-t\,x_i/x_{i+1}}{1-x_i/x_{i+1}}(s_i-1)+t^{\frac{1}{2}} \quad (i=1,\dots,n-1)$$

$\frac{t x_i - x_{i+1}}{x_i - x_{i+1}}$

$t=1$ s_0

$$T_0 = t^{-\frac{1}{2}} \frac{1-t\,q\,x_n/x_1}{1-q\,x_n/x_1}(s_0-1)+t^{\frac{1}{2}}$$

$$x_i/x_{i+1} = x^{\alpha_i} \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i=1,\dots,n-1)$$

$$q\,x_n/x_1 = x^{\alpha_0} \qquad \alpha_0 = \delta - \varepsilon_1 + \varepsilon_n$$

$$q = x^{\delta}$$

null root

affine simple root

$s_i^2=1$

(0) $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$

$s_i^{-1} = s_i$ $\quad T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ $\quad (i=0,1,\dots,n-1)$

(1) $T_i T_j = T_j T_i \quad (j \not\equiv i, i\pm1)$

(2) $T_i T_j T_i = T_j T_i T_j \quad (j \equiv i\pm1)$

(3) $\omega T_0 = T_{n-1}\omega$

$\omega T_i = T_{i-1}\omega \quad (i=1,\dots,n-1)$

defining relations of the affine Hecke algebra

③ q-Dunkl operators $Y_1, \dots, Y_n$ $\longleftrightarrow$ $\tau_1, \dots, \tau_n$

(Cherednik operators)

$$Y_1 = T_1 \cdots T_{n-1}\, \omega$$

$$Y_2 = T_2 \cdots T_{n-1}\, \omega\, T_1^{-1}$$

$$\vdots$$

$$Y_n = \omega\, T_1^{-1} \cdots T_{n-1}^{-1}$$

$$\begin{cases} \tau_1 = s_1 \cdots s_{n-1}\, \omega \\ \tau_2 = s_2 \cdots s_{n-1}\, \omega\, s_1 \\ \vdots \\ \tau_n = \omega\, s_1 \cdots s_{n-1} \end{cases}$$

$\omega(x_1) = q x_n$

$\omega(x_i) = x_{i-1}$

$i = 2, \dots, n$

Theorem $Y_i Y_j = Y_j Y_i \quad (i, j \in \{1, \dots, n\})$ ← defining relations

Exercise

$\mathbb{C}[Y^{\pm 1}] \subset H(\tilde{W})$ commutative subalgebra

$Y = (Y_1, \dots, Y_n)$ $\quad \simeq \mathbb{C}[\xi^{\pm 1}] \quad \xi = (\xi_1, \dots, \xi_n)$

$$H(W) = \mathbb{C}\langle T_1, \dots, T_{n-1} \rangle = \bigoplus_{w \in W} \mathbb{C} T_w$$

commutative subalgebra

$$H(\tilde{W}) = \mathbb{C}\langle T_0, T_1, \dots, T_{n-1}; \omega^{\pm 1} \rangle = \mathbb{C}[Y^{\pm 1}] \otimes H(W)$$

$$= \bigoplus_{w \in W} \mathbb{C}[Y^{\pm 1}] T_w$$

(Bernstein)

$$ZH(\tilde{W}) = \mathbb{C}[Y^{\pm 1}]^W = \{ f(Y) \mid f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W \}$$

center of $H(\tilde{W})$

$$Y = (Y_1, \dots, Y_n)$$

symmetric Laurent

$$\xi = (\xi_1, \dots, \xi_n)$$

$w = s_{i_1} \cdots s_{i_\ell}$ reduced decomposition (shortest)

$\ell = \ell(w)$

$T_w = T_{i_1} \cdots T_{i_\ell}$

Iwahori-Matsumoto.

⑤ From q-Dunkl operators to Macdonald-Ruijsenaars operators

$$ZH(\tilde{W}) = \mathbb{C}[Y^{\pm 1}]^W \longrightarrow \mathcal{D}_{q,x} = \mathbb{C}(x)[\tau^{\pm 1}]$$

$$\Psi \qquad\qquad \Psi$$

$$A_x = f(Y) \longmapsto L_x = L_x^f$$

$\mathcal{D}_{q,x}[W]$

$$A_x = f(Y) = \sum_{\substack{\mu \in P, \\ w \in W}} a_{\mu w}(x)\, \tau^{\mu} w$$

set $w = 1$

$$L_x = L_x^f = \sum_{\substack{\mu \in P \\ w \in W}} a_{\mu w}(x)\, \tau^{\mu}$$

If $\varphi(x)$ is a symmetric function

$$A_x \varphi(x) = L_x \varphi(x)$$

q-diff-refl. op q-diff op

Theorem $e_r(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W$ elementary symmetric function

$$L_x^{e_r} = t^{-\frac{1}{2}(n-1)r} D_x^{(r)} \quad (r=1,\dots,n)$$

$$e_r(Y)\varphi(x) = t^{-\frac{1}{2}(n-1)r} D_x^{(r)} \varphi(x)$$

← symmetric function

$$e_r(Y)P_\lambda(x) = e_r(t^\rho q^\lambda) P_\lambda(x).$$

$e_r(t^\rho q^\lambda)$

$$\rho = \frac{1}{2}(n-1, n-3, \dots, -n+1) = \delta - \frac{n-1}{2}(1,\dots,1)$$

In general,

$$f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W \Longrightarrow f(Y)P_\lambda(x) = L_x^f P_\lambda(x) = f(t^\rho q^\lambda) P_\lambda(x)$$

$\lambda \in \mathcal{P}_m$ (or P_+)

To summarize:

$$ZH(\tilde{W}) = \mathbb{C}[Y^{\pm 1}]^W \xrightarrow{\sim} \mathbb{C}[D_x^{(1)}, \dots, D_x^{(n)}, (D_x^{(1)})^{-1}]$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f(Y) & \longmapsto & L_x^f \end{array}$$

The commutative ring of Macdonald-Ruijsenaars operators arise from the center of the extended affine Hecke algebra (= the ring of symmetric Dunkl operators)

⑪ Double affine Hecke algebra and Cherednik involution

$$DH(\widetilde{W}) = \mathbb{C}\langle x_1^{\pm 1}, \dots, x_n^{\pm 1}; T_1, \dots, T_{n-1}; Y_1^{\pm 1}, \dots, Y_n^{\pm 1} \rangle$$

t-deformation

$$\mathbb{C}[x^{\pm 1}; \tau^{\pm 1}][W] = \mathbb{C}\langle x_1^{\pm 1}, \dots, x_n^{\pm 1}; s_1, \dots, s_{n-1}; \tau_1^{\pm 1}, \dots, \tau_n^{\pm 1} \rangle$$

q-Heisenberg algebra

$$\tau^{\mu} x^{\lambda} = q^{\langle \lambda, \mu \rangle} x^{\lambda} \tau^{\mu}$$

$$\begin{cases} \phi(ab) = \phi(b)\phi(a) \\ \phi^2 = 1 \end{cases}$$

$\exists!$ $\phi : DH(\widetilde{W}) \to DH(\widetilde{W})$: anti-involution

$$\phi(x_i) = Y_i^{-1}, \quad \phi(Y_i) = x_i^{-1}$$

$$\phi(T_i) = T_i \quad (i = 1, 2, \dots, n-1)$$

- $\langle \quad \rangle : DH(\widetilde{W}) \to \mathbb{C}$: expectation value

$$\langle A \rangle = A(1)\Big|_{x=t^{-\rho}} \quad : \quad \langle \phi(A) \rangle = \langle A \rangle$$

- scalar product : $\langle \, , \, \rangle : DH(\widetilde{W}) \times DH(\widetilde{W}) \to \mathbb{C}$

$$\langle A, B \rangle = \langle \phi(A) B \rangle \, , \quad \boxed{\langle A, B \rangle = \langle B, A \rangle}$$

Apply this to Macdonald polynomials

$$\langle P_\lambda(x), P_\mu(x) \rangle = \langle P_\mu(x), P_\lambda(x) \rangle \qquad (\lambda, \mu \in \mathcal{P}_n)$$

$$\Longrightarrow \quad \frac{P_\lambda(t^\rho q^\mu)}{P_\lambda(t^\rho)} = \frac{P_\mu(t^\rho q^\lambda)}{P_\mu(t^\rho)}$$

self-duality
(evaluation symmetry)