Symmetries of Painlevé equations: Lecture 9 (B1)

by M. Noumi [April 16, 2021]

• **References**:

[1] M. Noumi: Painlevé Equations through Symmetry, Translations of Mathematical Monographs **223**, American Mathematical Society, 2004. x+156 pp.

[2] K. Kajiwara, M. Noumi and Y. Yamada: Geometric aspects of Painlevé equations, Topical Review, J. Phys. A: Math. Theor. **50** (2017), 073001 (164 pp.)

1 An example: $P_{\rm II}$

In order to describe some characteristic features of Painlevé equations, we begin by an example of the second Painlevé differential equation $P_{\rm II}$.

1.1 Hamiltonian representation of $P_{\rm II}$

We consider the second order nonlinear differential equation

$$P_{\rm II} = P_{\rm II}(b): \qquad q'' = 2q^3 + tq + b - \frac{1}{2} \tag{1.1}$$

for the unknown function $q = q(t)$, called $P_{\rm II}$, where $' = d/dt$ and $b \in \mathbb{C}$. This equation can be equivalently written as a system of differential equations for two unkown functions $q = q(t)$ and $p = p(t)$

$$H_{\rm II} = H_{\rm II}(b): \qquad q' = p - q^2 - \frac{t}{2}, \quad p' = 2qp + b, \tag{1.2}$$

where $q = y$. If we set

$$H = H(p, q; t) = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - bq = \frac{1}{2}p\left(p - 2q^2 - t\right) - bq, \tag{1.3}$$

equation (1.2) is expressed as the Hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q} \tag{1.4}$$

with a time dependent Hamiltonian H.

1.2 Some special solutions

For some special values of $b \in \mathbb{C}$, $H_{\mathrm{II}}(b)$ has "elementary" special solutions.

- **Case $b = \frac{1}{2}$:** $H_{\mathrm{II}}(\frac{1}{2})$ has a simple rational solution $(q, p) = (0, \frac{t}{2})$.
- **Case $b = 0$:** $H_{\mathrm{II}}(0)$ has a 1-paramter family of solutions expressible by *Hermite functions*.

Suppose $b = 0$:

$$H_{\mathrm{II}}(0): \qquad q' = p - q^2 - \frac{t}{2}, \quad p' = 2qp \tag{1.5}$$

If we set $p = 0$, this equation is reduced to the *Riccati equation*

$$q' = -q^2 - \frac{t}{2}. \tag{1.6}$$

By the Hopf-Cole transformation $q = u'/u$, it can be rewritten into the linear differential equation (equivalent to Airy's equation)

$$u'' + \frac{t}{2}u = 0. \tag{1.7}$$

Taking two linearly independent solutions $\varphi_0(t)$, $\varphi_1(t)$ of (1.7), we obtain particular solutions

$$q = \frac{c_0\varphi_0'(t) + c_1\varphi_1'(t)}{c_0\varphi_0(t) + c_1\varphi_1(t)}, \quad p = 0 \qquad (c_0, c_1 \in \mathbb{C}; (c_0, c_1) \neq (0, 0)) \tag{1.8}$$

of $H_{\mathrm{II}}(0)$ parametrized by $[c_0 : c_1] \in \mathbb{P}^1$.

Other elementary solutions?

1.3 Fundamental Bäcklund transformations s and r

There are changes of dependent variables that transform any generic solution of H_{II} to another. Such transformations are called *Bäcklund transformations*.

Supposing that a pair of functions (q, p) satisfies $H_{\mathrm{II}}(b)$, we define

$$\widetilde{q} = q + \frac{b}{p}, \quad \widetilde{p} = p. \tag{1.9}$$

Then one can compute the equation to be satisfied by $(\widetilde{q}, \widetilde{p})$ as follows:

$$\begin{aligned} \widetilde{q}' &= q' - \frac{b}{p^2}p' = p - q^2 - \frac{t}{2} - \frac{b}{p^2}(2qp + b) \\ &= p - \left(q + \frac{b}{p}\right)^2 - \frac{t}{2} = \widetilde{p} - \widetilde{q}^2 - \frac{t}{2} \\ \widetilde{p}' &= p' = 2qp + b = 2\left(q + \frac{b}{p}\right)p - b = 2\widetilde{q}\widetilde{p} - b. \end{aligned} \tag{1.10}$$

This means that $(\widetilde{q}, \widetilde{p})$ is a solution of $H_{\mathrm{II}}(-b)$. In other words, the Hamiltonian system H_{II} is invariant under the change of dependent variables and the parameter

$$s: \qquad \widetilde{q} = q + \frac{b}{p}, \quad \widetilde{p} = p, \quad \widetilde{b} = -b. \tag{1.11}$$

Similarly, one can verify that the change of variables

$$\widetilde{q} = -q, \quad \widetilde{p} = -p + 2q^2 + t \tag{1.12}$$

transforms any solution of $H_{\mathrm{II}}(b)$ to a solution $H(1-b)$. We call this transformation as

$$r: \qquad \widetilde{q} = -q, \quad \widetilde{p} = -p + 2q^2 + t, \quad \widetilde{b} = 1 - b. \tag{1.13}$$

These s and r are the two fundamental Bäcklund transformations of H_{II}.

For example, applying these transformations to the rational solution $(q, p; b) = (0, \frac{t}{2}; \frac{1}{2})$ of H_{II}, we obtain

$$\left(0, \frac{t}{2}; \frac{1}{2}\right) \quad \stackrel{s}{\to} \quad \left(\frac{1}{t}, \frac{t}{2}; -\frac{1}{2}\right) \quad \stackrel{r}{\to} \quad \left(-\frac{1}{t}, \frac{t^3+4}{2t^2}; \frac{3}{2}\right). \tag{1.14}$$

Starting from the *seed solution* $(q, p; b) = (0, \frac{t}{2}; \frac{1}{2})$, we can use Bäcklund transformations to generate an infinite number of rational solutions at the paramter values $b \in \frac{1}{2} + \mathbb{Z}$.

Remark: Let $\mathcal{K} = \mathbb{C}(q, p, t, b)$ be the field of rational functions in (q, p, t, b) and define a derivation $\delta : \mathcal{K} \to \mathcal{K}$ by

$$\delta(q) = p - q^2 - \frac{t}{2}, \quad \delta(p) = 2qp + b, \quad \delta(t) = 1, \quad \delta(b) = 0. \tag{1.15}$$

The pair $(\mathcal{K}, \delta)$ is the *differential field* associated with the Hamiltonian system H_{II}. Then a Bäcklund transformation of H_{II} is interpreted as a field automorphism $w : \mathcal{K} \to \mathcal{K}$ is that commutes with the derivation δ. W define two automorphisms $s, r : \mathcal{K} \to \mathcal{K}$ by

$$\begin{array}{lllll} s: & s(q) = q + \dfrac{b}{p}, & s(p) = p, & s(t) = t, & s(b) = -b, \\ r: & r(q) = -q, & r(p) = -p + 2q^2 + t, & r(t) = t, & r(b) = 1 - b. \end{array} \tag{1.16}$$

Then, the computations presented as above imply that s and r commute with the derivation δ. Note also that $s^2 = 1$ and $r^2 = 1$. On the b-line, s and r represent the reflections with respect to the origin $b = 0$ and $b = \frac{1}{2}$, respectively. The group $\widetilde{W} = \langle s, r \rangle \subset \mathrm{Aut}_\delta(\mathcal{K})$ is the *extended affine Weyl group* of type A_1. Since $sr(b) = b + 1$, the Bäcklund transformation $w = (sr)^n$ $(n \in \mathbb{Z})$ transforms a generic solution (q, p) with parameter b to a solution $(w(q), w(p))$ with paramter $w(b) = b + n$.

1.4 Classical solutions of H_{II}

Theorem A:
(1) When $b \in \mathbb{Z}$, $H_{\mathrm{II}}(b)$ has a 1-paramter family of solutions which are expressible by the Airy function and its derivatives.
(2) When $b \in \frac{1}{2} + \mathbb{Z}$, $H_{\mathrm{II}}(b)$ has a rational solution.

These solutions are obtained by Bäcklund transformations from those at $b = 0$ and $b = \frac{1}{2}$, respectively. Also, it is known that any other solution of H_{II} is *very transcendental.*

The rational solutions at $b \in \frac{1}{2} + \mathbb{Z}$ has a remarkable factorization property.

Theorem B: There are two sequences of polynomials $P_n = P_n(t)$ and $Q_n = Q_n(t)$ $(n \in \mathbb{Z})$ in t such that the rational solution at $b = \frac{1}{2} - n$ is expressed in the form

$$q = \frac{nQ_n}{P_n P_{n+1}}, \quad p = \frac{P_{n-1}P_{n+1}}{2P_n^2}, \quad b = \frac{1}{2} - n \qquad (n \in \mathbb{Z}). \tag{1.17}$$

Table 1: Generating Rational Solutions

b	q	p
$\vdots$	$\vdots$	$\vdots$
$-\frac{7}{2}$	$\vdots$	$\dfrac{(t^3+4)(t^{10}+60t^7+11200t)}{2(t^6+20t^3-80)^2}$
$-\frac{5}{2}$	$\dfrac{3(t^8+8t^5+160t^2)}{(t^3+4)(t^6+20t^3-80)}$	$\dfrac{t(t^6+20t^3-80)}{2(t^3+4)^2}$
$-\frac{3}{2}$	$\dfrac{2(t^3-2)}{t(t^3+4)}$	$\dfrac{t^3+4}{2t^2}$
$-\frac{1}{2}$	$\dfrac{1}{t}$	$\dfrac{t}{2}$
$\frac{1}{2}$	0	$\dfrac{t}{2}$
$\frac{3}{2}$	$-\dfrac{1}{t}$	$\dfrac{t^3+4}{2t^2}$
$\frac{5}{2}$	$-\dfrac{2(t^3-2)}{t(t^3+4)}$	$\dfrac{t(t^6+20t^3-80)}{2(t^3+4)^2}$
$\frac{7}{2}$	$-\dfrac{3(t^8+8t^5+160t^2)}{(t^3+4)(t^6+20t^3-80)}$	$\dfrac{(t^3+4)(t^{10}+60t^7+11200t)}{2(t^6+20t^3-80)^2}$
$\vdots$	$\vdots$	$\vdots$

Here, P_n and Q_n are monic polynomials with integer coefficients and satisfy

$$\deg P_n = \frac{n(n-1)}{2}, \quad P_n = P_{1-n}; \quad \deg Q_n = n^2-1, \quad Q_n = Q_{-n}. \tag{1.18}$$

These polynomials P_n are called the *Yablonski-Vorobiev polynomials*:

$$\begin{aligned} &P_1 = 1, \quad P_2 = t, \quad P_3 = t^3+4, \quad P_4 = t^6+20t^3-80, \quad P_5 = t^{10}+60t^7+11200t, \ \dots \\ &Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = t^3-2, \quad Q_3 = t^8+8t^5+160t^2, \ \dots \end{aligned} \tag{1.19}$$

2 Symmetric form of P_{IV}

We now describe the symmetry of the fourth Painlevé equation

$$P_{\mathrm{IV}}: \qquad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y} \tag{2.1}$$

for $y = y(t)$, $' = d/dt$, from a slightly different point of view.

2.1 Symmetric form

We consider the following system of ordinary differential equations for three dependent variables $f_0,\ f_1,\ f_2$ with parameters $\alpha_0,\alpha_1,\alpha_2$:

$$S_{\rm IV}:\quad \begin{cases} f_0' = f_0(f_1-f_2)+\alpha_0, \\ f_1' = f_1(f_2-f_0)+\alpha_1, \\ f_2' = f_2(f_0-f_1)+\alpha_2. \end{cases} \tag{2.2}$$

When $\alpha_0=\alpha_1=\alpha_2$, (2.2) reduces to the *Lotka-Volterra equation* (competition model for three species.) Since

$$(f_0+f_1+f_2)' = \alpha_0+\alpha_1+\alpha_2 = k \quad \text{(constant)}, \tag{2.3}$$

we have $f_0+f_1+f_2 = kt+c$. Assuming that $k\neq 0$, we normalize the variables so that $k=1$, $c=0$:

$$\alpha_0+\alpha_1+\alpha_2=1, \quad f_0+f_1+f_2=t. \tag{2.4}$$

In the following, we regard the indices for f_j and α_j as elements of $\mathbb{Z}/3\mathbb{Z}$. Then equation (2.2) has a manifest rotation symmetry π of order 3:

$$\pi: \quad \pi(f_j)=f_{j+1}, \quad \pi(\alpha_j)=\alpha_{j+1} \qquad (j\in\mathbb{Z}/3\mathbb{Z}); \qquad \pi^3=1. \tag{2.5}$$

Under the normalization condition (2.4), the system of differential equations (2.2) is equivalent to the fourth Painlevé equation $P_{\rm IV}$. Since $f_0=t-f_1-f_2$, we can eliminate f_0:

$$f_1' = f_1(f_1+2f_2-t)+\alpha_1, \quad f_2' = f_2(t-2f_1-f_2)+\alpha_2. \tag{2.6}$$

From these two equations, together with the one obtained from the second by differentiation, we can eliminate f_1 and f_1'. The resulting equation for $q=f_2$ is given by

$$q'' = \frac{1}{2q}(q')^2+\frac{3}{2}q^3-2tq^2+\left(\frac{t^2}{2}+\alpha_0-\alpha_1\right)q-\frac{\alpha_2^2}{2q}, \tag{2.7}$$

which is identified with $P_{\rm IV}$ above (by rescaling q and t).

2.2 Some special solutions

If we impose the symmetry condition $\alpha_0=\alpha_1=\alpha_2$ and $f_0=f_1=f_2$, we obtain a rational solution

$$(\alpha_0,\alpha_1,\alpha_2;f_0,f_1,f_2)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3};\frac{t}{3},\frac{t}{3},\frac{t}{3}\right) \tag{2.8}$$

(the fixed point of π). On the other hand, when $\alpha_0=0$, the equation $f_0'=f_0(f_1-f_2)$ for f_0 is satisfied by $f_0=0$. Since $\alpha_1+\alpha_2=1$ and $f_1+f_2=t$, the remaining equations give rise to a single Riccati equation

$$f_1' = f_1(t-f_1)+\alpha_1 \tag{2.9}$$

for f_1. This Riccati equation is transformed by the change of variables $f_1=u'/u$ to the linear differential equation

$$u''-tu'-\alpha_1 u=0 \tag{2.10}$$

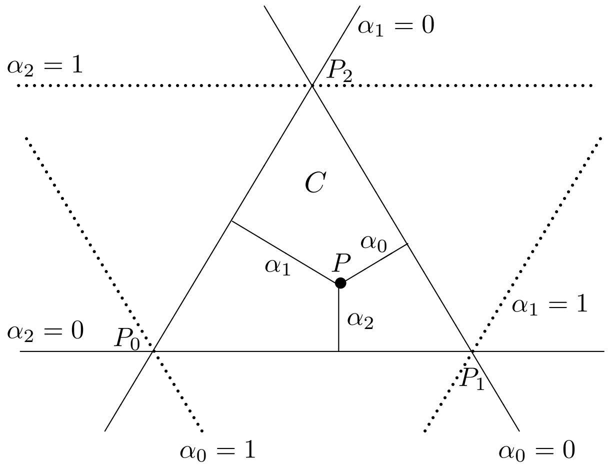


Figure 1: Triangular Coordinate System

equivalent to *Hermite's differential equation*. This means that, when $\alpha_0 = 0$, the system (2.2) has a 1-parameter family of solutions $(f_0, f_1, f_2) = (0, \frac{\varphi'}{\varphi}, t - \frac{\varphi'}{\varphi})$ which are expressible in terms of Hermite functions.

Note that the parameter space for $(\alpha_0, \alpha_1, \alpha_2)$ is the plane $\alpha_0 + \alpha_1 + \alpha_2 = 1$ in $\mathbb{C}^3$ endowed with the triangular coordinate system. In this picture, the rational solution (2.8) is located at the barycenter of the triangle $\Delta P_0 P_1 P_2$. Also, we know that along the three lines $\alpha_j = 0$ $(j = 0, 1, 2)$ bounding the triangle, there appear Riccati solutions expressible in terms of Hermite functions. At the three vertices P_0, P_1, P_2 where two lines intersect, there arise three simple rational solutions

$$(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (1, 0, 0 : t, 0, 0), \quad (0, 1, 0 : 0, t, 0), \quad (0, 0, 1 : 0, 0, t). \tag{2.11}$$

On this parameter space, one can consider three reflections (mirror images) s_0, s_1, s_2 with respect to the three lines $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, and the rotation π by 120° around the barycenter of $\Delta P_0 P_1 P_2$. The reflections s_j and the rotation π act as follows on the field $\mathbb{C}(\alpha_0, \alpha_1, \alpha_2)$ of rational functions on the parameter space:

$$\begin{array}{c|ccc} & \alpha_0 & \alpha_1 & \alpha_2 \\ \hline s_0 & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 \\ s_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 \\ s_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 \\ \hline \pi & \alpha_1 & \alpha_2 & \alpha_0 \end{array} \tag{2.12}$$

The group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$, generated by s_j and π, is called the *extended affine Weyl group of type* A_2. As an abstract group, $\widetilde{W}$ can be defined by the fundamental relations

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad \pi s_j = s_{j+1} \pi, \quad \pi^3 = 1, \tag{2.13}$$

where $j \in \mathbb{Z}/3\mathbb{Z}$. [1]

[1] It is sometimes more convenient to exclude the condition $\pi^3 = 1$ from the set of fundamental relations.

Table 2: Bäcklund Transformations for the Symmetric Form of $P_{\rm IV}$

	α_0	α_1	α_2	f_0	f_1	f_2
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	$\alpha_2+\alpha_0$	f_0	$f_1+\dfrac{\alpha_0}{f_0}$	$f_2-\dfrac{\alpha_0}{f_0}$
s_1	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	$f_0-\dfrac{\alpha_1}{f_1}$	f_1	$f_2+\dfrac{\alpha_1}{f_1}$
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$f_0+\dfrac{\alpha_2}{f_2}$	$f_1-\dfrac{\alpha_2}{f_2}$	f_2
π	α_1	α_2	α_0	f_1	f_2	f_0

2.3 Fundamental Bäcklund transformations

Returning to the symmetric form

$$S_{\rm IV}: \quad f_j' = f_j(f_{j+1}-f_{j+2})+\alpha_j \quad (j=0,1,2), \quad f_0+f_1+f_2=t, \tag{2.14}$$

of $P_{\rm IV}$, we define three dependent variables g_0, g_1, g_2 by

$$g_0=f_0, \quad g_1=f_1+\frac{\alpha_0}{f_0}, \quad g_2=f_2-\frac{\alpha_0}{f_0}. \tag{2.15}$$

The one can directly verify that g_0, g_1, g_2 satisfy

$$\begin{cases} g_0' = g_0(g_1-g_2)-\alpha_0 \\ g_1' = g_1(g_2-g_0)+\alpha_1+\alpha_0 \\ g_2' = g_2(g_0-g_1)+\alpha_2+\alpha_0. \end{cases} \tag{2.16}$$

Hence we obtain a Bäcklund transformation

$$(\alpha_0,\alpha_1,\alpha_3;f_0,f_1,f_2) \ \rightarrow \ (\beta_0,\beta_1,\beta_2;g_0,g_1,g_2) \tag{2.17}$$

with the change of parameters

$$\beta_0=-\alpha_0, \quad \beta_1=\alpha_1+\alpha_0, \quad \beta_2=\alpha_2+\alpha_0. \tag{2.18}$$

Note that the transformation of parameters $(\alpha_0,\alpha_1,\alpha_2)\rightarrow(\beta_0,\beta_1,\beta_2)$ is nothing but the action of the relfection s_0 with respect to the line α_0. For this reason, we use the same symbol s_0 for the Bäcklund transformation defined above. By the rotation of indices, we obtain the fundamental Bäcklund transformations s_0,s_1,s_2 together with π.

Theorem: The change of variables s_0,s_1,s_2 and π specified by Table 2 define Bäcklund transformations for the symmetric form (2.2) of $P_{\rm IV}$. Furthermore, they satisfy the fundamental realtions

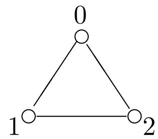
$$s_j^2=1, \quad (s_js_{j+1})^3=1, \quad \pi s_j = s_{j+1}\pi \quad (j\in\mathbb{Z}/3\mathbb{Z}), \quad \pi^3=1, \tag{2.19}$$

of the extended affine Weyl group $\widetilde{W}=\langle s_0,s_1,s_2,\pi\rangle$ of type A_2.

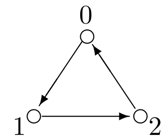
This means that the action of the exteded affine Weyl group on the parameter space lifts to the level of the dependent variables f_j of the differential system S_{IV}.

Remark: We define two integer matrices $A = \big(a_{i,j}\big)_{i,j=0}^{2}$ and $U = \big(u_{i,j}\big)_{i,j=0}^{2}$ as follows:

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$



$$U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad (2.20)$$



The matrix A is the *Cartan matrix* of the affine root system of type $A_2^{(1)}$, and U defines an orientation on the corresponding Dynkin diagram. With these matrices, the Bäcklund transformations s_0, s_1, s_2 in Table 2 are simply described as

$$s_i(\alpha_j) = \alpha_j - \alpha_i\, a_{ij}, \qquad s_i(f_j) = f_j + \frac{\alpha_i}{f_i}\, u_{ij} \qquad (i, j \in \mathbb{Z}/3\mathbb{Z}). \tag{2.21}$$

Table 3: The Six Painlevé Equations

$$
\begin{aligned}
P_{\mathrm{I}}: \quad & y'' = 6y^2 + t \\
P_{\mathrm{II}}: \quad & y'' = 2y^3 + ty + \alpha \\
P_{\mathrm{III}}: \quad & y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{\mathrm{IV}}: \quad & y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_{\mathrm{V}}: \quad & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' \\
& \qquad + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1} \\
P_{\mathrm{VI}}: \quad & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& \qquad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}
$$

In this list, $y = y(t)$ is the dependent variable and $' = d/dt$ stands for the derivative with respect to the independent variable t. The symbols $\alpha, \beta, \ldots$ are parameters.

Table 4: The Hypergeometric Equation and its Confluences

$$
\begin{aligned}
\text{Airy:} \quad & u'' - t\,u = 0 \\
\text{Bessel:} \quad & u'' + \frac{1}{t}\,u' + \left(1 - \frac{\alpha^2}{t^2}\right)u = 0 \\
\text{Hermite:} \quad & u'' - 2\,t\,u' + 2\,\alpha\,u = 0 \\
\text{Kummer:} \quad & u'' + \left(\frac{\gamma}{t} - 1\right)u' - \frac{\alpha}{t}\,u = 0 \\
\text{Gauss:} \quad & u'' + \frac{\gamma - (\alpha + \beta + 1)t}{t(1-t)}\,u' - \frac{\alpha\beta}{t(1-t)}\,u = 0
\end{aligned}
$$