《Symmetries of Painlevé equations》

Lecture 9 (B1)
2021/04/16

§1: An example: P_{II}

$$P_{II} = P_{II}(b): \quad q'' = 2q^3 + tq + b - \frac{1}{2}$$

$$\begin{cases} t: \text{independent variable} \\ q = q(t): \text{dependent variable} \quad ' = \frac{d}{dt} \\ b \in \mathbb{C} \end{cases}$$

1° Hamiltonian representation

$$H_{II} = H_{II}(b): \quad \begin{cases} q' = p - q^2 - \frac{t}{2} \\ p' = 2qp + b \end{cases}$$

q, p: dependent variables

$$H = H(q, p; \underline{\underline{t}}) = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - bq$$

$$\frac{\partial H}{\partial p} = p - q^2 - \frac{t}{2}, \quad \frac{\partial H}{\partial q} = -2qp - b$$

$$H_{II} \Leftrightarrow \underbrace{q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}}$$

Hamiltonian system

↑ time dependent

<u>Remark</u>. $H = H(q, p; t)$

$$h(t) = H(q(t), p(t); t) \qquad \underline{\underline{h'}} = \frac{\partial H}{\partial q} \cdot q' + \frac{\partial H}{\partial p} p' + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0.$$

H_{II} : $h' + \frac{\partial H}{\partial t} = -\frac{1}{2}p(t)$

<u>τ function</u> $\underbrace{h(t) = \frac{\tau'(t)}{\tau(t)}}$, $\underline{\underline{\tau(t)}} = \exp\left(\int^t h(t)\,dt\right)$

2° Some special solutions

$$H_{II}: \quad q' = p - q^2 - \frac{t}{2}, \quad p' = 2qp + b$$

(A) $q = 0$: $p = \frac{t}{2}$, $p' = \frac{1}{2}$ $\quad (q, p) = (0, \frac{t}{2})$ is a solution of $H_{II}(\frac{1}{2})$

(B) $p = 0$: $b = 0$ $\quad q' = -q^2 - \frac{t}{2}$ $\quad$ Riccati equation

$H_{II}(0)$ $\quad q = \frac{u'}{u} \Rightarrow u'' + \frac{t}{2} u = 0$ $\quad$ Airy's equation

Hopf-Cole

$\varphi_0(t), \varphi_1(t)$

$$u(t) = c_0 \varphi_0(t) + c_1 \varphi_1(t)$$

$$q(t) = \frac{c_0 \varphi_0'(t) + c_1 \varphi_1'(t)}{c_0 \varphi_0(t) + c_1 \varphi_1(t)} \qquad [c_0 : c_1] \in \mathbb{P}^1 \qquad (c_0, c_1) \neq (0, 0)$$

3° Fundamental Bäcklund transformation

$$H_{II}: \quad q' = p - q^2 - \frac{t}{2}, \quad p' = 2qp + b$$

(III) (S)

$$\tilde{q} = q + \frac{b}{p}, \quad \tilde{p} = p$$

$$\tilde{q}' = q' - \frac{b}{p^2} p' = p - q^2 - \frac{t}{2} - \frac{b}{p^2}(2qp + b)$$

$$= p - \left(q + \frac{b}{p}\right)^2 - \frac{t}{2} = \tilde{p} - \tilde{q}^2 - \frac{t}{2}$$

$$\tilde{p}' = p' = 2qp + b = 2\left(q + \frac{b}{p}\right)p - b = 2\tilde{q}\tilde{p} - b$$

$(\tilde{q}, \tilde{p})$: is a solution of $H_{II}(-b)$ $\quad \tilde{b} = -b$

(S) $s: \quad \tilde{q} = q + \frac{b}{p}, \quad \tilde{p} = p, \quad \tilde{b} = -b.$ $\qquad s^2 = id$

$$(q, p; b) \longrightarrow (\tilde{q}, \tilde{p}; \tilde{b})$$

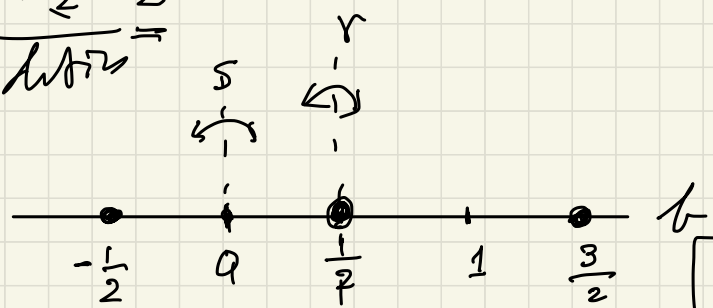
(R) (r)

$$r: \quad \tilde{q} = -q, \quad \tilde{p} = -p + 2q^2 + t, \quad \tilde{b} = 1 - b \qquad r^2 = id$$

<u>example</u>

(A) $\underbrace{(0, \frac{t}{2}; \frac{1}{2})}_{\text{seed solution}} \xrightarrow{s} (\frac{1}{t}, \frac{t}{2}; -\frac{1}{2}) \xrightarrow{r} (-\frac{1}{t}, \frac{t^3+4}{2t^2}; \frac{3}{2})$

$\boxed{r \circ s}$



$b \longrightarrow b+1$

$\boxed{b \in \frac{1}{2} + \mathbb{Z}}$

(B) $(\frac{n'}{n}, 0; 0) \dashrightarrow b \in \mathbb{Z}$

<u>Remark</u> $K = \mathbb{C}(q, p; t; b)$ $\qquad \delta(fg) = \delta(f)g + f\delta(g)$

$\delta = ' : K \longrightarrow K$: derivation

H_{II} : $\delta(q) = p - q^2 - \frac{t}{2}$, $\delta(p) = 2qp + b$ $\quad \delta(t) = 1$, $\delta(b) = 0$

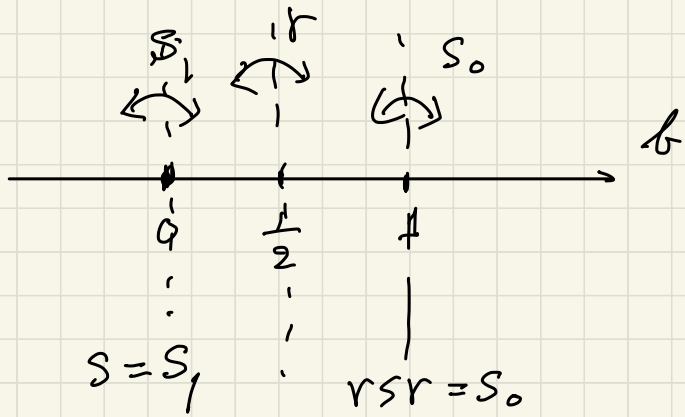
$$\delta = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial}{\partial t}$$ Hamiltonian vector field

(K, δ) : differential field representing H_{II}

$s, r : K \longrightarrow K$: field automorphism

$$\begin{cases} s(q) = q + \frac{b}{p}, \ s(p) = p, \ s(t) = t, \ s(b) = -b \\ r(q) = -q, \ r(p) = -p + 2q^2 + t, \ r(t) = t, \ r(b) = 1 - b \end{cases}$$

s, r : commute with δ. $\langle s, r \rangle \subseteq \mathrm{Aut}_\delta(K)$



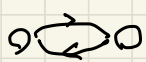
$s(b) = -b$

$r(b) = 1 - b$

$$\tilde{W} = \langle s_0, s_1 ; \underset{\sim}{r} \rangle \subseteq \mathrm{Aut}_{\mathbb{Z}}(K)$$

extended affine Weyl group of type $\underline{\underline{A_1^{(1)}}}$

s_0 s_1



$A_1^{(1)}$

4° Classical solutions

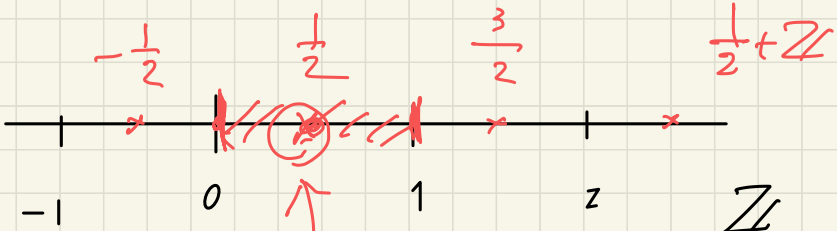
<u>Theorem A:</u>

(1) $b \in \mathbb{Z}$; $H_{II}(b)$ has a 1-parameter family of solutions expressible by the Airy function and its derivative. $\mathbb{P}^1$

(2) $b \in \frac{1}{2} + \mathbb{Z}$; $H_{II}(b)$ has a rational solution. hypergeometric

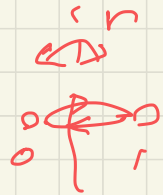
These are obtained by Bäcklund transformations of those at $b=0$, $b=\frac{1}{2}$. <u>Umemura</u>

"The other solutions of H_{II} are <u>very transcendental</u>!



$\mathbb{Z}$

reflection hyperplanes, wall



fixed point of

$\left\{ \begin{array}{l} \text{Dynkin diagram} \\ \text{automorphism} \end{array} \right.$

(11) rational solutions at $\underline{b \in \frac{1}{2}+\mathbb{Z}}$

§2: Symmetric form of P_{IV}

$$P_{IV}: \; y'' = \underline{\frac{1}{2}(y')^2 + \frac{3}{2}y^3} + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}.$$

1° Symmetric form of P_{IV}

Z/34

$$S_{IV}: \begin{cases} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 \end{cases}$$

- $f_j = f_j(t)$, $' = \frac{d}{dt}$, $\alpha_j \in \mathbb{C}$
- $\alpha_0 = \alpha_1 = \alpha_2 = 0$ <u>Lotka-Volterra</u> ; <u>Kac-van Moerbeke</u>

$$S_{IV}: \quad f_j' = f_j(f_{j+1} - f_{j-1}) + \alpha_j \qquad j = 0,1,2 \in \mathbb{Z}/3\mathbb{Z}$$

$$(*)\begin{cases} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 \end{cases} \qquad (\alpha_j \in \mathbb{C})$$

$$\boxed{g = f_2}$$

$$(f_0 + f_1 + f_2)' = \alpha_0 + \alpha_1 + \alpha_2 = k \ (\text{const}) \qquad k \neq 0$$

$$f_0 + f_1 + f_2 = kt + c$$

$$(k = 1)$$

Normalize $\begin{cases} \alpha_0 + \alpha_1 + \alpha_2 = 1 \\ f_0 + f_1 + f_2 = t \end{cases}$

$$f_0 = t - f_1 - f_2$$

$$q'' = \frac{1}{2q}(q')^3 + \frac{3}{2}q^3 - 2tq^2 + \left(\frac{t^2}{2} + \alpha_0 - \alpha_1\right)q - \frac{\alpha_2^2}{2q}.$$

$\sim P_{IV}$