Symmetries of Painlevé equations: Lecture 10 (B2)

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3 τ Functions

3.1 Poisson structure and Hamiltonian representation

Recall that our symmetric form of P_{IV} is defined by

$$S_{\mathrm{IV}}: \quad f_0' = f_0(f_1 - f_2) + \alpha_0, \quad f_1' = f_1(f_2 - f_0) + \alpha_1, \quad f_2' = f_2(f_0 - f_1) + \alpha_2 \tag{3.1}$$

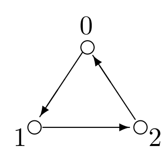
with normalizaion $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $f_0 + f_1 + f_2 = t$.

Denoting by $\mathcal{K} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ the field of rational functions in α_j and f_j $(j = 0, 1, 2)$, we introduce a Poisson bracket $\{\ ,\ \}$ on $\mathcal{K}$ by

$$\{\varphi, \psi\} = \sum_{i,j=0}^{2} \frac{\partial \varphi}{\partial f_i} u_{i,j} \frac{\partial \psi}{\partial f_j}, \quad (i, j \in \{0, 1, 2\}), \quad U = \big(u_{i,j}\big)_{i,j=0}^{2} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \tag{3.2}$$

so that $\{f_i, f_j\} = u_{ij}$, $\{\alpha_i, \alpha_j\} = \{\alpha_i, f_j\} = 0$.

$$\begin{aligned} &\{f_j, f_j\} = 0 \qquad (j = 0, 1, 2) \\ &\{f_0, f_1\} = 1, \quad \{f_1, f_0\} = -1 \\ &\{f_1, f_2\} = 1, \quad \{f_2, f_1\} = -1 \\ &\{f_2, f_0\} = 1, \quad \{f_0, f_2\} = -1 \end{aligned} \tag{3.3}$$



By a *Poisson bracket*, we mean a $\mathbb{C}$-bilinear form $\{\ ,\ \}: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ satisfying the conditions

$$\begin{aligned} &(1) \quad \{f, f\} = 0, \quad \{g, f\} = -\{f, g\}, \\ &(2) \quad \{fg, h\} = \{f, h\}\, g + f\, \{g, h\}, \quad \{f, gh\} = \{f, g\}\, h + g\, \{f, h\}, \\ &(3) \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \end{aligned} \tag{3.4}$$

Then the system of differential equations (3.1) is expressed as

$$f_0' = \{H, f_0\} + 1, \quad f_1' = \{H, f_1\}, \quad f_2' = \{H, f_2\} \tag{3.5}$$

in terms of the Hamiltonian

$$H = f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2. \tag{3.6}$$

From this expression, we see that, for any $\varphi \in \mathcal{K}$, the derivation $' : \mathcal{K} \to \mathcal{K}$ defining S_{IV} is described as

$$\varphi' = \{H, \varphi\} + \frac{\partial \varphi}{\partial f_0}. \tag{3.7}$$

We set $p = f_1$, $q = f_2$ so that

$$f_0 = t - p - q, \quad f_1 = p, \quad f_2 = q, \quad f_0 + f_1 + f_2 = t. \tag{3.8}$$

Then we have

$$\{p, q\} = 1, \quad \{p, p\} = \{q, q\} = 0, \quad \{p, t\} = \{q, t\} = \{t, t\} = 0. \tag{3.9}$$

Note also that $\mathcal{K} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; p, q; t)$. In terms of the coordinates $(q, p; t)$, we have

$$\{\varphi, \psi\} = \frac{\partial \varphi}{\partial p}\frac{\partial \psi}{\partial q} - \frac{\partial \varphi}{\partial q}\frac{\partial \psi}{\partial p} \tag{3.10}$$

and

$$H = (t - q - p)pq + \alpha_2 p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t. \tag{3.11}$$

From this, we see that the symmetric form of P_{IV} is equivalent to the Hamiltonian system

$$q' = \{H, q\} = \frac{\partial H}{\partial p}, \quad p' = \{H, p\} = -\frac{\partial H}{\partial q}. \tag{3.12}$$

Namely

$$H_{\mathrm{IV}}: \quad q' = q(t - q - 2p) + \alpha_2, \quad p' = p(2q + p - t) + \alpha_1. \tag{3.13}$$

We also remark that the Bäcklund transformations s_i $(i = 0, 1, 2)$ are described as

$$s_i(f_j) = f_j + \frac{\alpha_i}{f_i}\{f_i, f_j\} \qquad (i, j \in \{0, 1, 2\}) \tag{3.14}$$

in terms of the Poisson bracket. Furthermore, one can verify that, for any $w \in \widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$, the corresponding Bäcklund transformation $w : \mathcal{K} \to \mathcal{K}$ defines a canonical transformation in the sense that

$$w(\{\varphi, \psi\}) = \{w(\varphi), w(\psi)\} \qquad (\varphi, \psi \in \mathcal{K}). \tag{3.15}$$

This means that the extended Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ is realized as a group of birational canonical transformations which preserve the differential system S_{IV} invariant.

3.2 τ Functions

In order to recover the cyclic symmetry, we intoduce three Hamiltonians h_0, h_1, h_2 by $h_0 = H$, $h_1 = \pi(h_0)$, $h_2 = \pi(h_1)$:

$$\begin{aligned}
h_0 &= f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2, \\
h_1 &= f_0 f_1 f_2 + \frac{1}{3}(\alpha_2 - \alpha_0) f_1 + \frac{1}{3}(\alpha_2 + 2\alpha_0) f_2 - \frac{1}{3}(2\alpha_2 + \alpha_0) f_0, \\
h_2 &= f_0 f_1 f_2 + \frac{1}{3}(\alpha_0 - \alpha_1) f_2 + \frac{1}{3}(\alpha_0 + 2\alpha_1) f_0 - \frac{1}{3}(2\alpha_0 + \alpha_1) f_1.
\end{aligned} \tag{3.16}$$

We introduce three τ *functions* (τ variables) τ_0, τ_1, τ_2 as dependent variables such that

$$h_0 = \frac{\tau_0'}{\tau_0}, \quad h_1 = \frac{\tau_1'}{\tau_1}, \quad h_2 = \frac{\tau_2'}{\tau_2}. \tag{3.17}$$

Formally, we can write $h_i = (\log \tau_i)'$ and $\tau_i = \exp(\int h_i dt)$ $(i = 0, 1, 2)$. Then, by (3.16), we see directly that the f variables are recovered as

$$f_0 = h_2 - h_1 + \frac{t}{3} = \frac{\tau_2'}{\tau_2} - \frac{\tau_1'}{\tau_1} + \frac{t}{3}. \tag{3.18}$$

We now compute the differential equations to be satisfied by h_i. Since $H = h_0$, we have

$$h_0' = \{h_0, h_0\} + \frac{\partial h_0}{\partial f_0} = f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) \tag{3.19}$$

Combinig three formulas

$$h_0' = f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2), \quad h_1' = f_2 f_0 + \frac{1}{3}(\alpha_2 - \alpha_0), \quad h_2' = f_0 f_1 + \frac{1}{3}(\alpha_0 - \alpha_1), \tag{3.20}$$

one can verify that

$$(h_0 + h_1)' + (h_0 - h_1)^2 + \frac{t}{3}(h_0 - h_1) - \frac{2}{9}t^2 + \frac{1}{3}(\alpha_0 - \alpha_1) = 0. \tag{3.21}$$

Using $h_i = \tau_i'/\tau_i$, this equation can be rewritten into the bilinear equation

$$\tau_0''\tau_1 - 2\tau_0'\tau_1' + \tau_0\tau_1'' + \frac{t}{3}(\tau_0'\tau_1 - \tau_0\tau_1') - \Big(\frac{2}{9}t^2 - \frac{1}{3}(\alpha_0 - \alpha_1)\Big)\tau_0\tau_1 = 0, \tag{3.22}$$

namely,

$$\Big(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{1}{3}(\alpha_0 - \alpha_1)\Big)\tau_0 \cdot \tau_1 = 0 \tag{3.23}$$

in terms of the *Hirota derivatives*

$$D_t(f \cdot g) = f'g - fg', \quad D_t^2(f \cdot g) = f''g - 2f'g' + fg'', \quad \ldots. \tag{3.24}$$

In this way, the differential system S_{IV} is translated into the system of bilinear differential equations

$$\begin{aligned} &\Big(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{1}{3}(\alpha_0 - \alpha_1)\Big)\tau_0 \cdot \tau_1 = 0, \\ &\Big(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{1}{3}(\alpha_1 - \alpha_2)\Big)\tau_1 \cdot \tau_2 = 0, \\ &\Big(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{1}{3}(\alpha_2 - \alpha_0)\Big)\tau_2 \cdot \tau_0 = 0 \end{aligned} \tag{3.25}$$

for the τ functions τ_0, τ_1, τ_2. The f variables are recovered from the τ functions by

$$f_0 = \frac{\tau_2'}{\tau_2} - \frac{\tau_1'}{\tau_1} + \frac{t}{3}, \quad f_1 = \frac{\tau_0'}{\tau_0} - \frac{\tau_2'}{\tau_2} + \frac{t}{3}, \quad f_2 = \frac{\tau_1'}{\tau_1} - \frac{\tau_0'}{\tau_0} + \frac{t}{3}, \tag{3.26}$$

namely

$$f_0 = \frac{(D_t + \frac{t}{3})\tau_2 \cdot \tau_1}{\tau_2\tau_1}, \quad f_1 = \frac{(D_t + \frac{t}{3})\tau_0 \cdot \tau_2}{\tau_0\tau_2}, \quad f_2 = \frac{(D_t + \frac{t}{3})\tau_1 \cdot \tau_0}{\tau_1\tau_0}. \tag{3.27}$$

3.3 Lifting the Bäcklund transformations to the τ functions

It is *not* an obvious procedure to find Bäcklund transformations for the τ functions.

Noting that $h_j = \tau_j'/\tau = (\log \tau_j)'$, we investigate how the three hamiltonians transform by the action of $\widetilde{W}$. It is a direct computation, but the result is interesting:

$$s_i(h_j) = h_j \quad (i \neq j), \quad s_j(h_j) = h_j + \frac{\alpha_j}{f_j} \quad (i, j \in \{0, 1, 2\}). \tag{3.28}$$

Namely, h_j is invariant under s_i $(i \neq j)$ and transforms by s_j only. (This means that h_j and τ_j are in some way attached to the *fundamental weights* of the affine root system, while f_j correspond to the simple roots α_j.) Noting that $f_1 - f_2 = 2h_0 - h_1 - h_2$, from

$$\frac{f_0'}{f_0} = 2h_0 - h_1 - h_2 + \frac{\alpha_0}{f_0}, \quad s_0(h_0) = h_0 + \frac{\alpha_0}{f_0}, \tag{3.29}$$

we obtain

$$s_0(h_0) = \frac{f_0'}{f_0} + h_1 + h_2 - h_0. \tag{3.30}$$

This means that

$$s_0\Big(\frac{\tau_0'}{\tau_0}\Big) = \frac{f_0'}{f_0} + \frac{\tau_1'}{\tau_1} + \frac{\tau_2'}{\tau_2} - \frac{\tau_0'}{\tau_0}, \tag{3.31}$$

which would imply

$$s_0(\tau_0) = \text{const.}\, f_0\, \frac{\tau_1 \tau_2}{\tau_0}. \tag{3.32}$$

Theorem: Define the trasformations s_0, s_1, s_2 and π for τ functios by

$$s_i(\tau_j) = \tau_j \quad (i \neq j), \quad s_j(\tau_j) = f_j \frac{\tau_{j-1}\tau_{j+1}}{\tau_j}, \quad \pi(\tau_j) = \tau_{j+1} \qquad (i, j \in \mathbb{Z}/3\mathbb{Z}). \tag{3.33}$$

Then, these commute with the derivation $'$, providing Bäcklund transformations for τ functions. Furthermore, they satisfy the fundamental relations

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad \pi s_i = s_{i+1}\pi \quad (i \in \mathbb{Z}/3\mathbb{Z}), \quad \pi^3 = 1 \tag{3.34}$$

for $\widetilde{W}$, with respect to all the variables including τ functions.

Wer remark that the variables f_j are recovered from τ functions by

$$f_j = \frac{\tau_j\, s_j(\tau_j)}{\tau_{j-1}\tau_{j+1}} \qquad (j \in \mathbb{Z}/3\mathbb{Z}) \tag{3.35}$$

in terms of the fundamental Bäcklund transformations s_j. Recall that

$$s_1(f_2) = f_2 + \frac{\alpha_1}{f_1}, \quad s_2(f_1) = f_1 - \frac{\alpha_2}{f_2}. \tag{3.36}$$

In terms of the τ functions, these two equations can be rewritten as

$$\tau_1 s_1 s_2(\tau_2) - s_1(\tau_1) s_2(\tau_2) = \alpha_1 \tau_0^2, \quad s_1(\tau_1) s_2(\tau_2) - s_2 s_1(\tau_1)\tau_2 = \alpha_2 \tau_0^2 \tag{3.37}$$

Then, eliminating τ_0 we obtain the *Hirota-Miwa equation*

$$(\alpha_0 - 1)\, s_1(\tau_1)\, s_2(\tau_2) + \alpha_1\, s_2 s_1(\tau_1)\, \tau_2 + \alpha_2\, \tau_1\, s_1 s_2(\tau_2) = 0 \tag{3.38}$$

relating six τ functions placed at the vertices of a regular hexagon.

3.4 Computation of Bäcklund transformations

Let $w \in \widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ an element of the extended affine Weyl group, and set

$$\beta_j = w(\alpha_j) \in Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2, \qquad g_j = w(f_j) \in \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) \tag{3.39}$$

for each $j \in \mathbb{Z}/3\mathbb{Z}$. Then, from a generic solution $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ of S_{IV} we obtain a new soluiotn $(\beta_0, \beta_1, \beta_2; g_0, g_1, g_2)$ of S_{IV}.

• **Some examples:**

$$\begin{aligned}
s_1(f_2) &= f_2 + \frac{\alpha_1}{f_1} = \frac{f_1 f_2 + \alpha_1}{f_1} \\
s_2 s_1(f_2) &= f_2 + \frac{\alpha_1 + \alpha_2}{f_1 - \frac{\alpha_2}{f_2}} = \frac{f_2(f_1 f_2 + \alpha_1)}{f_1 f_2 - \alpha_2} \\
s_0 s_2 s_1(f_2) &= f_2 - \frac{a_0}{f_0} - \frac{2\alpha_0 + \alpha_1 + \alpha_2}{f_1 + \frac{\alpha_0}{f_0} + \frac{a_0 + a_2}{f_2 - \frac{\alpha_0}{f_0}}} \\
&= \frac{(f_0 f_2 - \alpha_0)\big(f_0^2 f_1 f_2 + (\alpha_0 + \alpha_1) f_0^2 - \alpha_0 f_0 (f_1 - f_2) - \alpha_0^2\big)}{f_0\big(f_0^2 f_1 f_2 - (\alpha_0 + \alpha_2) f_0^2 - \alpha_0 f_0 (f_1 - f_2) - \alpha_0^2\big)} \\
s_1(\tau_1) &= f_1 \frac{\tau_0 \tau_2}{\tau_1} \\
s_2 s_1(\tau_1) &= (f_1 f_2 - \alpha_2) \frac{\tau_0^2}{\tau_2} \\
s_0 s_2 s_1(\tau_1) &= \big(f_0^2 f_1 f_2 - (\alpha_0 + \alpha_1) f_0^2 - \alpha_0 f_0 (f_1 - f_2)) - \alpha_0^2\big) \frac{\tau_1^2 \tau_2}{\tau_0^2}
\end{aligned} \tag{3.40}$$

Theorem: For each $w \in \widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ and $j = 0, 1, 2$, there exists a unique polynomial $\phi_{w,j} \in \mathbb{Z}[\alpha_0, \alpha_1, \alpha_2, f_0, f_1, f_2]$ $(j = 0, 1, 2)$ such that

$$w(\tau_j) = \phi_{w,j}\, \tau_0^{k_0} \tau_1^{k_1} \tau_2^{k_2} \qquad (k_0, k_1, k_2 \in \mathbb{Z}), \tag{3.41}$$

where k_0, k_1, k_2 are determined from w and j. The Bäcklund transformations $w(f_j)$ of f_j are then expressed as

$$w(f_j) = \frac{w(\tau_j) \;\; w s_j(\tau_j)}{w(\tau_{j-1}) w(\tau_{j+1})} = \frac{\phi_{w,j} \;\; \phi_{w s_j, j}}{\phi_{w,j-1}\, \phi_{w,j+1}} \quad (j = 0, 1, 2). \tag{3.42}$$

This theorem explains the factorization property of the Bäcklund transformations. We call $\phi_{w,j}$ the ϕ *factor* for the τ function $w(\tau_j)$; this family of polynomials $\phi_{w,j}$ $(w \in \widetilde{W}, j = 0, 1, 2)$ is also called the τ *cocycle*. Note that $\phi_{w,j}$ satisfy

$$\phi_{1,j} = 1, \quad \phi_{s_i w, j} = s_i(\phi_{m,j})\, f_i^{k_i}, \quad \phi_{\pi w, j} = \pi(\phi_{w,j}); \tag{3.43}$$

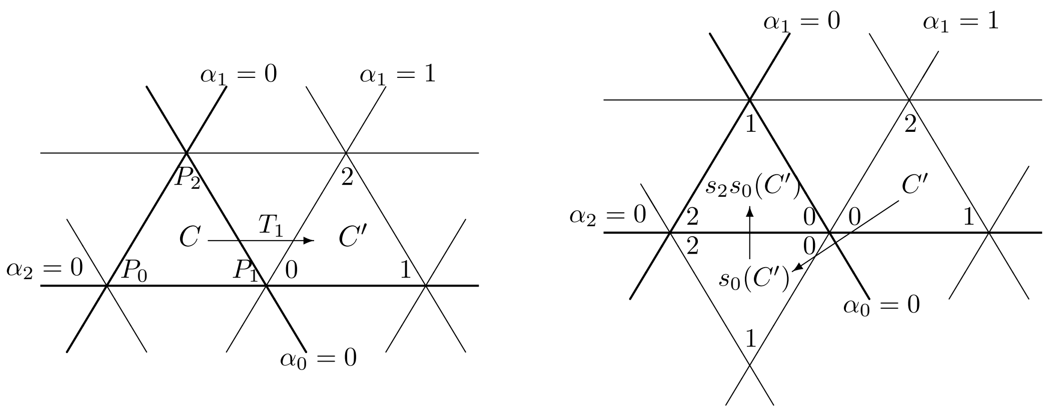
By this recursion, $\phi_{w,j}$ are determined as rational functions in f_0, f_1, f_2, but it is known that they are in fact polynomials.

3.5 Translations in the extended affine Weyl group

We remark that the extended affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ contains the commutative subgroup of *translations*. We denote by T_1, T_2, T_3 the translations with respect to the edges $\overline{P_0P_1}$, $\overline{P_1P_2}$, $\overline{P_2P_0}$ of the regular triangle $\Delta P_0P_1P_2$, respectively. They are expressed as

$$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2, \quad T_3 = s_2 s_1 \pi \tag{3.44}$$

in terms of the generators of the extended affine Weyl group $\widetilde{W}$.



(3.45)

By the fundamental relations of $\widetilde{W}$, one can directly verify that

$$T_i T_j = T_j T_i \quad (i, j = 1, 2, 3), \qquad T_1 T_2 T_3 = 1. \tag{3.46}$$

From this construction, we see that the extended Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ is expressed as the semi-direct product of $\langle s_1, s_2 \rangle = \mathfrak{S}_3$ ($s_1 = (12), s_2 = (24)$) and the abelian subgroup of translations:

$$\widetilde{W} = \mathfrak{S}_3 \ltimes T, \quad T = \left\{ T_1^l T_2^m T_3^n \mid l, m, n \in \mathbb{Z} \right\}; \quad \sigma T_j = T_{\sigma(j)} \sigma \quad (\sigma \in \mathfrak{S}_3; j = 1, 2, 3). \tag{3.47}$$

We now look at the action of $T_1 = \pi s_2 s_1$ on α_j, f_j and τ_j. (Since $\pi(T_1) = T_2$, $\pi(T_2) = T_3$, the corresponding formulas for T_2 and T_3 are obtained by the diagram rotation π.)

$$\begin{aligned}
&T_1(\alpha_0) = \alpha_0 + 1, \quad T_1(\alpha_1) = \alpha_1 - 1, \quad T_1(\alpha_2) = \alpha_2. \\
&T_1(f_0) = f_1 + \frac{\alpha_0}{f_0} - \frac{\alpha_2 + \alpha_0}{f_1 - \frac{\alpha_0}{f_0}} = \frac{f_0^2 f_1 f_2 - \alpha_0 f_0 f_1 + \alpha_0 f_2 f_0 - (\alpha_0 + \alpha_2) f_0^2 - \alpha_0^2}{f_0(f_0 f_2 - \alpha_0)}, \\
&T_1(f_1) = f_2 - \frac{\alpha_0}{f_0} = \frac{f_0 f_2 - \alpha_0}{f_0}, \qquad T_1(f_2) = f_0 + \frac{\alpha_0 + \alpha_2}{f_2 - \frac{\alpha_0}{f_0}} = \frac{f_0(f_0 f_2 + \alpha_2)}{f_0 f_2 - \alpha_0}. \\
&T_1(\tau_0) = \tau_1, \quad T_1(\tau_1) = (f_0 f_2 - \alpha_0) \frac{\tau_1^2}{\tau_0}, \quad T_1(\tau_2) = f_0 \frac{\tau_1 \tau_2}{\tau_0}.
\end{aligned} \tag{3.48}$$

We remark that this Bäcklund transformation T_1 can be regarded as the time evolution of a *discrete Painlevé equation*. By iterating T_1, we set

$$x_n = T_1^n(f_1), \quad y_n = T_1^n(f_0) \qquad (n \in \mathbb{Z}). \tag{3.49}$$

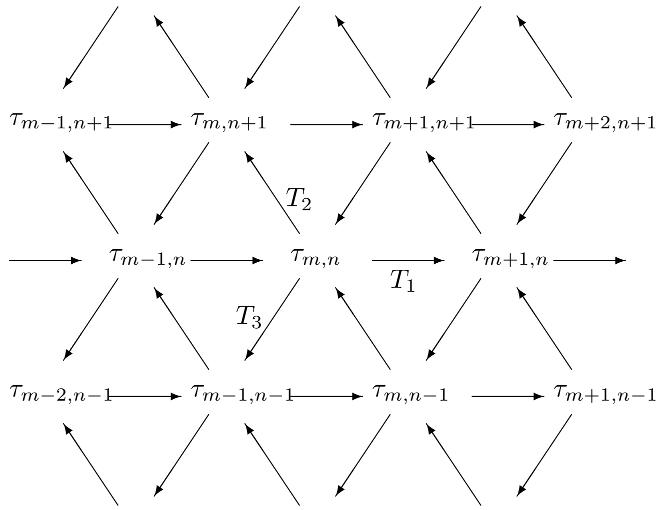


Figure 1: τ-Functions on the Lattice: $\tau_{m,n} = \tau_{m,n,0}$

Then from

$$T_1(f_1) = f_2 - \frac{\alpha_0}{f} = t - f_0 - f_1 - \frac{\alpha_0}{f_0}, \quad T_1^{-1}(f_0) = f_2 + \frac{\alpha_1}{f_1} = t - f_0 - f_1 + \frac{\alpha_1}{f_1} \tag{3.50}$$

we obtain

$$x_n + x_{n+1} = t - y_n - \frac{\alpha_0 + n}{y_n}, \quad y_{n-1} + y_n = t - x_n + \frac{\alpha_1 - n}{x_n} \qquad (n \in \mathbb{Z}). \tag{3.51}$$

This recurrence is (a version of) the *discret Painlevé equation* P_{II}; if fact, one can show that it passes to the Painlevé equation P_{II} by an appropriate continuum limit.

A more systematic way to understand the discrete symmetry of S_{IV} is to use the lattice τ functions on the triangular lattice.

$$\tau_{m,n} = T_1^m T_2^n(\tau_0) \quad (m, n \in \mathbb{Z}). \tag{3.52}$$

Note that $\tau_{0,0} = \tau_0$, $\tau_{1,0} = \tau_1$, $\tau_{1,1} = \tau_2$. The we have three types of bilinear equations for the lattice τ-functions.

(A) Bilinear differential equations of Hirota type:

$$\left(D_t^2 + \frac{t}{3} D_t - \frac{2t^2}{9} + \frac{\alpha_0 - \alpha_1 + 2m - n}{3} \right) \tau_{m,n} \cdot \tau_{m+1,n} = 0 \tag{3.53}$$

(B) Toda equation:

$$\tau_{m-1,n} \tau_{m+1,n} = \left(\frac{1}{2} D_t^2 - \frac{\alpha_0 - \alpha_1 + 2m - n - 1}{3} \right) \tau_{m,n} \cdot \tau_{m,n}. \tag{3.54}$$

(C) Hirota-Miwa equation:

$$(\alpha_0 + m - 1)\, \tau_{m,n-1}\, \tau_{m,n+1} + (\alpha_1 - m + n)\, \tau_{m+1,n+1}\, \tau_{m-1,n-1} + (\alpha_2 - n)\, \tau_{m-1,n}\, \tau_{m+1,n} = 0. \tag{3.55}$$

Note also that there exist the ϕ factors $\phi_{m,n} \in \mathbb{Z}[\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2]$ such that

$$\tau_{m,n} = \phi_{m,n}\, \tau_0^{1-m} \tau_1^{m-n} \tau_2^{n} \quad (m, n \in \mathbb{Z}). \tag{3.56}$$

The corresponding f variables are then expressed as

$$\begin{aligned}
g_0 = T_1^m T_2^n(f_0) &= \frac{\tau_{m,n}\tau_{m+2,n+1}}{\tau_{m+1,n}\tau_{m+1,n+1}} &= \frac{\phi_{m,n}\phi_{m+2,n+1}}{\phi_{m+1,n}\phi_{m+1,n+1}},\\
g_1 = T_1^m T_2^n(f_1) &= \frac{\tau_{m+1,n}\tau_{m,n+1}}{\tau_{m,n}\tau_{m+1,n+1}} &= \frac{\phi_{m+1,n}\phi_{m,n+1}}{\phi_{m,n}\phi_{m+1,n+1}},\\
g_2 = T_1^m T_2^n(f_2) &= \frac{\tau_{m+1,n+1}\tau_{m,n-1}}{\tau_{m,n}\tau_{m+1,n}} &= \frac{\phi_{m+1,n+1}\phi_{m,n-1}}{\phi_{m,n}\phi_{m+1,n}}.
\end{aligned} \tag{3.57}$$

The triple (g_0, g_1, g_2) of rational functions in f_0, f_1, f_2 gives a solution of S_{IV} with parameter values $(\beta_0, \beta_1, \beta_2) = (\alpha_0 + m, \alpha_1 - m + n, \alpha_2 - n)$.

3.6 Classical solutions of S_{IV}

Theorem:
(1) When $(\alpha_0, \alpha_1, \alpha_2)$ is the barycenter of an alcove (a triangle), S_{IV} has a rational solution obtained by a Bäcklund transformation from the seed solution $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{t}{3}, \frac{t}{3}, \frac{t}{3})$.
(2) When $(\alpha_0, \alpha_1, \alpha_2)$ is on the line $\alpha_j = n$ for some $j = 0, 1, 2$ and $n \in \mathbb{Z}$, S_{IV} has a 1-parameter family of solutions expressible ratioanlly by the Hermite functions and their derivatives.

It is also known that any other solution of S_{IV} is *very transcendental*.

If we specialize the ϕ factors $\phi_{m,n}$ to the rational solution $(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{t}{3}, \frac{t}{3}, \frac{t}{3})$, we obtain a family of polynomials

$$F_{m,n} = F_{m,n}(t) = \phi_{m,n}\Big|_{\alpha_j = \frac{1}{3}, f_j = \frac{t}{3}} \qquad (m, n \in \mathbb{Z}). \tag{3.58}$$

in t, which we are called the *Okamoto polynomials*. Then the rational solution with parameter values $(\alpha_0, \alpha_1, \alpha_2) = (\frac{1}{3} + m, \frac{1}{3} - m + n, \frac{1}{3} - n)$ is expressed as

$$(f_0, f_1, f_2) = \Big(\frac{F_{m,n}F_{m+2,n+1}}{F_{m+1,n}F_{m+1,n+1}}, \frac{F_{m+1,n}F_{m,n+1}}{F_{m,n}F_{m+1,n+1}}, \frac{F_{m+1,n+1}F_{m,n-1}}{F_{m,n}F_{m+1,n}}\Big), \tag{3.59}$$

in terms of Okamoto polynomials. This completely describe the factorization property of the rational solution at the barycenter of each alcove.

Remark: It is known that the ϕ factors $\phi_{m,n}$ for the τ functions $\tau_{m,n} = T_1^m T_2^n(\tau_0)$ has an explicit determinant formula of Jacobi-Trudi type (see [1]). Through this formula of Jacibi-Trudi type, one can identify each Okamoto polynomial $F_{m,n}(t)$ with a specialization of the Schur function $S_\lambda(t_1, t_2, \ldots)$ in the KP times $(t_1, t_2, \ldots)$ attached to a 3-core partition λ.