Symmetries of Painlevé equations: Lecture 11 (B3)

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4 Relation to integrable hierarchies

In general, there are several ways to interpret a Painlevé equation as the compatibility condition of a linear differential equation. In this section, we explain how the Painlevé equations P_{II}, P_{IV} as well as P_{V} are related with the modified KP hierarchy.

4.1 A Lax pair for P_{IV}

We consider a system of linear differential equations

$$x\partial_x \boldsymbol{u} = A(x,t)\boldsymbol{u}, \quad \partial_t \boldsymbol{u} = B(x,t)\boldsymbol{u}. \tag{4.1}$$

for a vector $\boldsymbol{u} = (u_1, \ldots, u_n)^{\mathrm{t}}$ of unknown functions $u_i = u_i(x,t)$ $(i = 1, \ldots, n)$ in two variables (x,t), where $\partial_x = \partial/\partial_x$ and $\partial_t = \partial/\partial t$. We assume that

$$A(x,t) = \big(a_{ij}(x,t)\big)_{i,j=1}^{n}, \quad B(x,t) = \big(b_{ij}(x,t)\big)_{i,j=1}^{n} \tag{4.2}$$

are $n\times n$ matrices of rational functions in x depending holomorphically on t. Then the compatibility (integrability, zero curvature) condition for (4.1) is given by

$$[x\partial_x - A(x,t), \partial_t - B(x,t)] = \partial_t A(x,t) - x\partial_x B(x,y) + [A(x,t), B(x,t)] = 0, \tag{4.3}$$

where $[\ ,\ ]$ denotes the commutator of matrices of differential operators. In general, this type of compatibility condition implies a system of nonlinear differential equations for the coefficients of $A(x,t)$, $B(x,t)$. In the following, we use the notations like $u_i(x) = u_i(x,t)$, $A(x) = A(x,t)$, $B(x) = B(x,t)$, $\ldots$ suppressing the dependence on t.

The symmteric form S_{IV} of the fourth Painlevé equation P_{IV} is relevant to the following system of linear differential equations. Setting $n = 3$, we consider 3×3 matrices $A(x)$, $B(x)$ of the form

$$\begin{aligned}
A(x) &= -\begin{bmatrix} \varepsilon_1 & f_1 & 1 \\ x & \varepsilon_2 & f_2 \\ f_0\, x & x & \varepsilon_3 \end{bmatrix} = -\begin{bmatrix} \varepsilon_1 & f_1 & 1 \\ 0 & \varepsilon_2 & f_2 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ f_0 & 1 & 0 \end{bmatrix} x, \\
B(x) &= \begin{bmatrix} v_1 & -1 & 0 \\ 0 & v_2 & -1 \\ -x & 0 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & -1 & 0 \\ 0 & v_2 & -1 \\ 0 & 0 & v_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} x.
\end{aligned} \tag{4.4}$$

In this case, we have

$$
\begin{aligned}
&\partial_t A(x) - x\partial_x B(x) + \big[A(x), B(x)\big] \\
&= \begin{bmatrix} -\varepsilon_1' & \varepsilon_1-\varepsilon_2+(v_1-v_2)f_1-f_1' & f_1-f_2+v_1-v_3 \\ (f_2-f_0+v_2-v_1)x & -\varepsilon_2' & \varepsilon_2-\varepsilon_3+(v_2-v_3)f_2-f_2' \\ (1+\varepsilon_3-\varepsilon_1+(v_3-v_1)f_0-f_0')x & (f_0-f_1+v_3-v_2)x & -\varepsilon_3' \end{bmatrix},
\end{aligned}
$$

where $' = \partial_t$, and hence, the compatibility condition is given by

$$
\begin{cases} \varepsilon_1' = 0 \\ \varepsilon_2' = 0 \\ \varepsilon_3' = 0 \end{cases}
\quad
\begin{cases} f_0' = (v_3-v_1)f_0 + (1+\varepsilon_3-\varepsilon_1) \\ f_1' = (v_1-v_2)f_1 + (\varepsilon_1-\varepsilon_2) \\ f_2' = (v_2-v_3)f_2 + (\varepsilon_2-\varepsilon_3) \end{cases}
\quad
\begin{cases} f_1 - f_2 = v_3 - v_1 \\ f_2 - f_0 = v_1 - v_2 \\ f_0 - f_1 = v_2 - v_3 \end{cases}
\tag{4.5}
$$

This system is equivalent to the symmetric form S_{IV}:

$$
\begin{cases} f_0' = (f_1-f_2)f_0 + \alpha_0 \\ f_1' = (f_2-f_0)f_1 + \alpha_1 \\ f_2' = (f_1-f_2)f_2 + \alpha_2 \end{cases}
\quad \text{with parameters} \quad
\begin{cases} \alpha_0 = 1+\varepsilon_3-\varepsilon_1 \\ \alpha_1 = \varepsilon_1-\varepsilon_2 \\ \alpha_2 = \varepsilon_2-\varepsilon_3 \end{cases}
\tag{4.6}
$$

The v variables here can be parametrized by hamiltonians as

$$
v_1 = h_1 - h_0, \quad v_2 = h_2 - h_1, \quad v_3 = h_0 - h_2. \tag{4.7}
$$

4.2 Relation to the n-reduced modified KP hierarchy

Generalizing this construction of the symmetric form S_{IV} for P_{IV}, one can formulate a class of nonlinear differential equations associated with $n \times n$ linear differential systems for arbitrary $n = 1, 2, 3, \ldots$.

In the following, we denote by $\mathbb{C}(\!(x^{-1})\!) = \mathbb{C}[[x^{-1}]][x]$ the ring of formal Laurent series in x whose exponents are bounded above, and by $\mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!))$ the algebra of $n \times n$ matrices with coefficients in $\mathbb{C}(\!(x^{-1})\!)$; $\mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!))$ is the space of all matrices

$$
A(x) = \sum_{k \in \mathbb{Z}} A_k x^k, \quad A_k \in \mathrm{Mat}(n; \mathbb{C}) \quad (k \in \mathbb{Z}), \tag{4.8}
$$

such that $A_k = 0$ for $k \gg 0$. We also denote by $\mathrm{GL}(n; \mathbb{C}(\!(x^{-1})\!))$ the group of invertible matrices in $\mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!))$. (We regard $\mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!))$ and $\mathrm{GL}(n; \mathbb{C}(\!(x^{-1})\!))$ as formal versions of the loop algebra of $\mathfrak{gl}_n = \mathrm{Mat}(n; \mathbb{C})$ and the loop group of $\mathrm{GL}(n; \mathbb{C})$.) We introduce the matrix (*cyclic element*)

$$
\Lambda(x) = \sum_{i=1}^{n-1} E_{i,i+1} + E_{1,n}x = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ x & 0 & \ldots & 0 & 0 \end{bmatrix} \in \mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!)) \tag{4.9}
$$

so that

$$\Lambda(x)^k = \sum_{i=1}^{n-k} E_{i,i+k} + \sum_{j=1}^{k} x E_{n-k+j,j} \quad (k=0,1,\ldots,n-1), \qquad \Lambda(x)^n = x I_n. \tag{4.10}$$

(Note that $\Lambda(x)^k$ ($k \in \mathbb{Z}$) are generators of the Heisenberg algebra). Then each matrix $A(x) \in \mathrm{Mat}(n; \mathbb{C}(\!(x^{-1})\!))$ can be expressed in the form

$$A(x) = \sum_{k=-\infty}^{m} \mathrm{diag}(\boldsymbol{a}_k)\Lambda(x)^k, \qquad \boldsymbol{a}_k \in \mathbb{C}^n \quad (k \in \mathbb{Z}, \quad k \le m) \tag{4.11}$$

for some $m \in \mathbb{Z}$, where $\mathrm{diag}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i E_{ii}$ denotes the diagonal matrix attached to $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. In the following, we use the notation

$$A(x)_{\ge 0} = \sum_{k \ge 0} \mathrm{diag}(\boldsymbol{a}_k)\Lambda(x)^k, \quad A(x)_{<0} = \sum_{k<0} \mathrm{diag}(\boldsymbol{a}_k)\Lambda(x)^k, \tag{4.12}$$

for the nonnegative part and the negative part of $A(x)$ respectively, so that $A(x) = A(x)_{\ge 0} + A(x)_{<0}$. In this convention, the nonegative part of $A(x)$ takes the form

$$A(x)_{\ge 0} = \begin{bmatrix} a_1^{(0)} & a_1^{(1)} & \ldots & \ldots & a_1^{(n-1)} \\ 0 & a_2^{(0)} & a_2^{(1)} & \ldots & a_2^{(n-2)} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n-1}^{(0)} & a_{n-1}^{(1)} \\ 0 & 0 & \ldots & 0 & a_n^{(0)} \end{bmatrix} + \begin{bmatrix} a_1^{(n)} & a_1^{(n+1)} & \ldots & a_1^{(2n-1)} \\ a_2^{(n-1)} & a_2^{(n)} & \ldots & a_2^{(2n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(1)} & a_n^{(2)} & \ldots & a_n^{(n)} \end{bmatrix} x + \cdots \tag{4.13}$$

with $\boldsymbol{a}_k = (a_1^{(k)}, \ldots, a_n^{(k)})$. In particular, the matrix of the coefficients of x^0 is upper triangular.

We now consider a sequence of $n \times n$ matrices of the form

$$B_k(x) = \sum_{r=0}^{k-1} \mathrm{diag}(\boldsymbol{b}_r^{(k)})\Lambda(x)^r + \Lambda(x)^k \qquad (k = 1, 2, \ldots) \tag{4.14}$$

In what follows, we use a sequence of m time variables $t = (t_1, t_2, \ldots, t_m)$.[1)] Regarding the coefficients $\boldsymbol{b}_r^{(k)}$ as vectors of functions in t, we consider the system of linear partial differential equations

$$\partial_{t_k} \boldsymbol{u} = B_k(x)\boldsymbol{u} \quad (k = 1, 2, \ldots, m) \tag{4.15}$$

for the vector $\boldsymbol{u} = (u_1, \ldots, u_n)^{\mathrm{t}}$ of unknown functions $u_i(x) = u(x,t)$ ($i = 1, \ldots .n$). The compatibility condition $\big[\partial_{t_k} - B_k(x), \partial_{t_l} - B_l(x)\big] = 0$ ($k, l = 1, 2, \ldots, m$) for this linear problem is called the *Zakharov-Shabat equations* for the *n-reduced modified KP hierarchy*: namely,

$$\partial_{t_l} B_k(x) - \partial_{t_k} B_l(x) + \big[B_k(x), B_l(x)\big] = 0 \qquad (k, l = 1, 2, \ldots, m). \tag{4.16}$$

[1)] Formally, one can introduce an infinite number of time variables $t = (t_1, t_2, \ldots)$.

Taking another $n \times n$ matrix $A(x)$ of the form

$$A(x) = \sum_{r=0}^{m} \mathrm{diag}(\boldsymbol{a}_r)\Lambda(x)^r, \tag{4.17}$$

we consider the system of linear partial differential equations

$$x\partial_x \boldsymbol{u} = A(x)\boldsymbol{u}, \qquad \partial_{t_k}\boldsymbol{u} = B_k(x)\boldsymbol{u} \quad (k = 1, \ldots, m) \tag{4.18}$$

in $(x, t) = (x, t_1, \ldots, t_m)$, regarding $\boldsymbol{a}_r$ as functions in t as well, In this system, we also need to assume the compatibility condition $\big[x\partial_x - A(x), \partial_{t_k} - B_k(x)\big] = 0$ $(k = 1, \ldots, m)$ besides the Zakharov-Shabat equations (4.16):

$$\partial_{t_k} A(x) - x\partial_x B_k(x) + \big[A(x), B_k(x)\big] = 0 \qquad (k = 1, \ldots, m). \tag{4.19}$$

These compatibility conditions (4.16) and (4.19) together can be thought of as defining a nonlinear system of partial differential equations of "Painlevé type". We remark that, in the previous setting of the Lax pair for the symmetric form S_{IV}, we considered the case where $n = 3$ and $m = 2$ and

$$\boldsymbol{a_0} = -(\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \boldsymbol{a_1} = -(f_1, f_2, f_0), \quad \boldsymbol{a_2} = -(1, 1, 1). \tag{4.20}$$

In the context of modified KP hierarchy, this procedure of adding the linear equation $x\partial_x \boldsymbol{u} = A(x)\boldsymbol{u}$ in x is interpreted as the *similarity reduction* as we will see below.

In the theory of modified KP hierarchy, we usually assume that the linear problem (4.15) has a system of fundamental formal solutions of the form

$$\Psi(x) = W(x)\, e^{\sum_{k=1}^{m} t_k \Lambda(x)^k}, \quad W(x) = 1 + \sum_{r=1}^{\infty} \mathrm{diag}(\boldsymbol{w}_r)\Lambda(x)^{-r} \in \mathrm{GL}(n; \mathbb{C}[[x^{-1}]]) \tag{4.21}$$

The system of equations to be satisfied by the *wave operator* $W(x)$,

$$\partial_{t_k} W(x) = B_k(x)W(x) - W(x)\Lambda(x)^k \quad (k = 1, 2, \ldots), \tag{4.22}$$

is sometimes called the *Sato equation.* We define the *Lax operator* $L(x)$ by

$$L(x) = W(x)\Lambda(x)W(x)^{-1}, \quad L(x) = \Lambda(x) + \sum_{r=1}^{\infty} \mathrm{diag}(\boldsymbol{u}_r)\Lambda(x)^{1-r} \tag{4.23}$$

Then, $L(x)$ is characterized by the *Lax equation*

$$\partial_{t_k} L(x) = \big[B_k(x), L(x)\big], \quad B_k(x) = (L(x)^k)_{\geq 0} \quad (k = 1, 2, \ldots). \tag{4.24}$$

We also remark that, if $W(x)$ is obtained through the *Riemann-Hilbert-Birkhoff decomposition*

$$e^{\sum_k t_k \Lambda(x)^k}\, \overset{\circ}{C}(x) = W(x)^{-1} Z(x), \quad Z(x) = \sum_{r=0}^{\infty} \mathrm{diag}(\boldsymbol{z}_r)\Lambda(x)^r, \tag{4.25}$$

for a given a matrix $\overset{\circ}{C}(x) \in \mathrm{GL}(n; \mathbb{C}(\!(x^{-1})\!))$ which does not depend on t, then $W(t)$ gives rises to a solution to (4.22) with $B_k(x) = (L(x)^k)_{\geq 0}$ $(k = 1, 2, \ldots)$.

Note that $\Lambda(x)$ satisfies the commutation relation

$$\big[nx\partial_x + \mathrm{diag}(\rho), \Lambda(x)\big] = \Lambda(x), \quad \rho = (n-1, n-2, \ldots, 0). \tag{4.26}$$

In view of

$$L(x) = W(x)\Lambda(x)W(x)^{-1} = \Psi(x)\Lambda(x)\Psi(x)^{-1}, \tag{4.27}$$

we define the *M-operator* $M(x)$ by

$$M(x) = \Psi(x)\big(nx\partial_x + \mathrm{diag}(\rho)\big)\Psi(x)^{-1} \tag{4.28}$$

Then we have

$$\big[M(x), L(x)\big] = L(x), \quad \partial_{t_k} M(x) = \big[B_k(x), M(x)\big] \quad (k = 1, 2, \ldots). \tag{4.29}$$

Furthermore, the matrix

$$\Phi(x) = \Psi(x)x^{-\mathrm{diag}(\rho/n)} = W(x)e^{\sum_k t_k \Lambda(x)^k} x^{-\mathrm{diag}(\rho/n)} \tag{4.30}$$

satisfies the system of differential equations

$$M(x)\Phi(x) = 0, \quad \partial_{t_k}\Phi(x) = B_k(x)\Phi(x) \quad (k = 1, \ldots, m). \tag{4.31}$$

By (4.21), (4.28), we see that the M-operator is decomposed as $M(x) = M(x)_{\geq 0} + M(x)_{<0}$, where

$$\begin{aligned} &M(x)_{\geq 0} = nx\partial_x + \rho - \textstyle\sum_{k=1}^m k t_k B_k(x), \quad B_k(x) = (L(x)^k)_{\geq 0}, \\ &M(x)_{<0} = -\big[nx\partial_x + \mathrm{diag}(\rho), W(x)\big]W(x)^{-1} + \textstyle\sum_{k=1}^m k t_k B_k^c(x), \quad B_k^c(x) = -(L(x)^k)_{<0}. \end{aligned} \tag{4.32}$$

From this expression, we see that the negative part $M(x)_{<0}$ vanishes if and only if the coefficients of the wave operator $W(x)$ satisfy the *similarity condition*

$$\textstyle\sum_{k=1}^m k t_k \partial_{t_k}(\boldsymbol{w}_r) = -r\boldsymbol{w}_r \quad (r = 1, 2, \ldots) \tag{4.33}$$

(the homogeneity condition in $t = (t_1, \ldots, t_m)$ with respect to the degrees $\deg t_k = k$ $(k = 1, 2, \ldots)$. We define the matrix $A(x)$ by

$$A(x) = -\mathrm{diag}(\tfrac{\rho}{n}) + \textstyle\sum_{k=1}^m \frac{k}{n} t_k B_k \tag{4.34}$$

so that $M(x)_{\geq 0} = n\big(x\partial_x - A(x)\big)$. Then under the similarity codition (4.33), the matrix

$$\Phi(x) = \Psi(x)x^{-\mathrm{diag}(\rho/n)} = W(x)e^{\sum_k t_k \Lambda(x)^k} x^{-\mathrm{diag}(\rho/n)} \tag{4.35}$$

satisfies the system of linear equations

$$x\partial_x\Phi(x) = A(x)\Phi(x), \quad \partial_{t_k}\Phi(x) = B_k(x)\Phi(x) \quad (k = 1, \ldots, m). \tag{4.36}$$

This means that the system of nonlinear equations

$$\big[x\partial_x - A(x), \partial_{t_k} - B_k(x)\big] = 0, \quad \big[\partial_{t_k} - B_k(x), \partial_{t_l} - B_l(x)\big] = 0 \quad (k, l = 1, \ldots, m) \tag{4.37}$$

of "Painlevé type" can be solved (has solutions obtained) by the similarity reduction of the n-reduced modified KP hierarchy.

In fact, the three cases of Painlevé equations P_{II}, P_{IV}, P_{V} are interpreted as similarity reduction of the n-reduced modified KP hierarchy (= modified Drinfeld-Sokolov hierarchy of type $A_{n-1}^{(1)}$) for $n = 2, 3, 4$ respectively. It is also known that P_{VI} can be interpreted by a version of the Drinfeld-Sokolov hierarchy of type $D_4^{(1)}$.