§2: Symmetric form of P_{IV}

§3: τ Functions

Lecture 11 (B3)
2021/04/30

$$(P_{IV}) \qquad q'' = \frac{1}{2q}(q')^2 + \frac{3}{2}q^3 - 2tq^2 + \left(\frac{t^2}{2} + \alpha_0 - \alpha_1\right)q - \frac{\alpha_2^2}{2q} \qquad (\alpha_0+\alpha_1+\alpha_2=1)$$

$$(H_{IV}) \quad \begin{cases} q' = q(t-q-2p) + \alpha_2 = \dfrac{\partial H}{\partial p} \\ p' = p(2q+p-t) + \alpha_1 = -\dfrac{\partial H}{\partial q} \end{cases} \qquad H = (t-p-q)pq + \alpha_2 p - \alpha_1 q + \frac{1}{3}(\alpha_1-\alpha_2)t$$

$\alpha_1 = 0;\ p = 0 \Rightarrow$ Riccati eq. $q' = q(t-q) + \alpha_2$

$\alpha_2 = 0,\ q = 0 \Rightarrow$ " $p' = p(p-t) + \alpha_1$

p : invariant divisor along $\alpha_1 = 0$

q : " $\alpha_2 = 0$

(i) Any other invariant divisors?

yes

(ii) $t-p-q$: is an invariant divisor along $1-\alpha_1-\alpha_2 = 0$

(Noumi-Okamoto ~1986)

$$\begin{cases} f_1 = p & : \text{invariant divisor along } \alpha_1 = 0 \\ f_2 = q & : \quad \text{"} \qquad\qquad\qquad \alpha_2 = 0 \\ f_0 = t - p - q & : \qquad\qquad\qquad \alpha_0 = 1 - \alpha_1 - \alpha_2 = 0 \end{cases}$$

$$h_0 = H$$

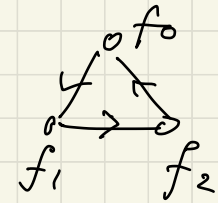
② How does P_{IV} appear if one uses the three invariant divisors as dependent variables? (~1996?)

$$\Rightarrow \quad (S_{IV}) \quad \begin{cases} f_0' = f_0(f_1 - f_2) + \alpha_0 \\ f_1' = f_1(f_2 - f_0) + \alpha_1 \\ f_2' = f_2(f_0 - f_1) + \alpha_2 \end{cases}$$

$$(f_0 + f_1 + f_2 = t, \ \alpha_0 + \alpha_1 + \alpha_2 = 1)$$

$$h_0 = f_0 f_1 f_2 + \frac{1}{3}(\alpha_1 - \alpha_2) f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2) f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2) f_2$$

$$\varphi' = \{H, \varphi\} + \frac{\partial \varphi}{\partial f_0}$$



(Noumi–Yamada)

Remark

Painlevé : Stockholm lectures

differential Galois theory for nonlinear differential equations

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(iii) H. Umemura (~1986)

irreducibility (transcendency of solutions) for Painlevé equations

in the language of modern mathematics

$$\left\{ \begin{array}{l} \text{Classification of } \left\{ \begin{array}{l} \text{algebraic solutions} \\ \text{invariant divisors} \end{array} \right. \\ \Longrightarrow \text{Any other solutions are "non-classical"} \end{array} \right.$$

- Umemura / Kishioka : irreducibility of P_I (1986)
- Classification of invariant divisors

  P_{II}, P_{IV} : Noumi-Okamoto P_{III} : Murata

  P_V, P_{VI} : Umemura-Watanabe (~ early 90's)
- Classification of algebraic solutions Dubrovin, Mazzocco, Boalch, Kitaev, ...

  $P_{II} \sim P_V$: Murata, ... P_{VI} : ~ Lisovyy-Tykhyy (2014)

④ Affine Weyl group symmetry

$$\widetilde{W} = \langle s_0, s_1, s_2; \pi \rangle : \quad s_i^2 = 1, \ (s_i s_{i+1})^3 = 1, \ \pi s_i = s_{i+1} \pi \quad (i \in \mathbb{Z}/3\mathbb{Z})$$

$$\curvearrowright \quad \mathcal{K} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) \subset \mathcal{L} = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2; \tau_0, \tau_1, \tau_2)$$

simple roots $\alpha_0, \alpha_1, \alpha_2$ — differential fields

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}$$

$$s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}$$

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad U = (u_{ij}) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$u_{ij} = \{ f_i, f_j \}$$

fundamental weights $\Lambda_0, \Lambda_1, \Lambda_2$

$$s_i(\tau_j) = \tau_j \quad (i \neq j)$$

$$s_j(\tau_j) = f_j \frac{\tau_{j-1} \tau_{j+1}}{\tau_j}$$

$$\Longleftarrow \quad s_i(h_j) = h_j \quad i \neq j$$

$$s_j(h_j) = h_j + \frac{\alpha_j}{f_j}$$

$$h_j = \frac{\tau_j'}{\tau_j}$$

$$= h_{j-1} + h_{j+1} - h_j + \frac{f_j'}{f_j}$$

4° Computations of Bäcklund transformations

- <u>Examples</u> $\Rightarrow$ summary pdf $\qquad \mathbb{Z}[\alpha; f]$

- <u>Theorem</u> $\forall w \in \widetilde{W}$ $\forall j \in \{0,1,2\}$ $\exists!$ $\phi_{w,j} = \phi_{w,j}(\alpha; f)$

  - $w(\tau_j) = \phi_{w,j}\, \tau_0^{k_0} \tau_1^{k_1} \tau_2^{k_2} \quad (k_i \in \mathbb{Z})$

  - $w(f_j) = \dfrac{w(\tau_j)\, w s_j(\tau_j)}{w(\tau_{j-1})\, w(\tau_{j+1})} = \dfrac{\phi_{w,j}\, \phi_{w s_j, j}}{\phi_{w,j-1}\, \phi_{w,j+1}}$

$$f_j = \frac{\tau_j\, s_j(\tau_j)}{\tau_{j-1}\, \tau_{j+1}}$$

$(\phi_{w,j})$ "ϕ factors"

"τ cocycle"

<u>Remark</u> $\phi_{w,j}$ are determined recursively by

$$\phi_{1,j}=1, \quad \phi_{s_i w,j} = s_i(\phi_{w,j}) f_i^{k_i}, \quad \phi_{\pi w,j} = \pi(\phi_{w,j})$$

as elements in $K = \mathbb{C}(\alpha; f)$.

However, it turns out that $\phi_{w,j} \in \mathbb{Z}[\alpha; f]$.

"special polynomials" associated PIV

$\Uparrow$

existence of <u>τ functions</u>

$\sim$ Laurent property in the theory of cluster algebras.

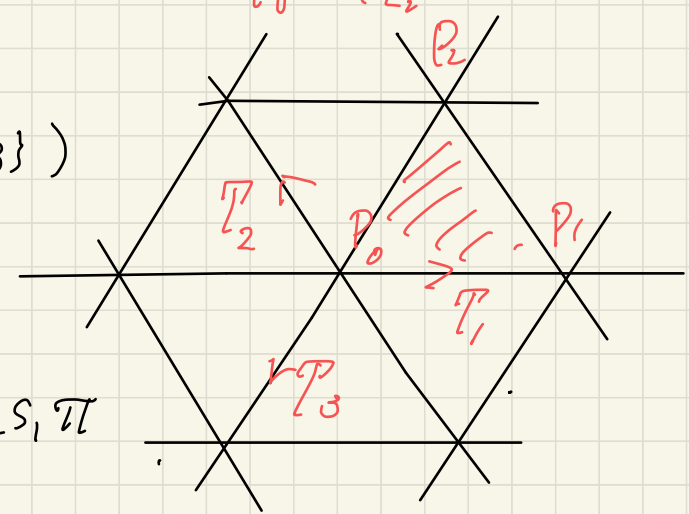
5° Translations in $\widetilde{W} = \langle s_0, s_1, s_2; \pi \rangle$

$\alpha_0 = 1 - \varepsilon_1 + \varepsilon_3$
$\alpha_1 = \varepsilon_1 - \varepsilon_2, \ \alpha_2 = \varepsilon_2 - \varepsilon_3$

$\widetilde{W} = \mathfrak{S}_3 \ltimes T \qquad \mathfrak{S}_3 = \langle s_1, s_2 \rangle \qquad s_1 = (12) \quad s_2 = (23)$

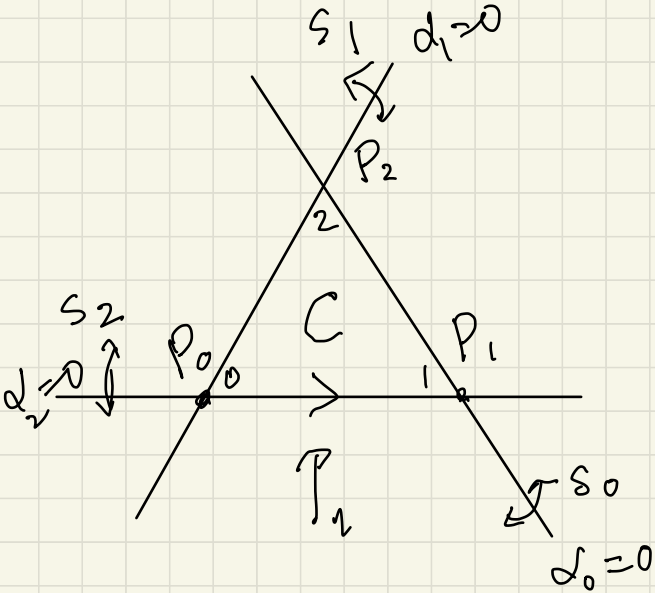
- $T = \{ T_1^{\ell} T_2^{m} T_3^{n} \,;\, \ell, m, n \in \mathbb{Z} \} = \{ T_1^{m} T_2^{n} \,;\, m, n \in \mathbb{Z} \}$
- $T_i T_j = T_j T_i, \ T_1 T_2 T_3 = 1$
- $\sigma T_j = T_{\sigma(j)} \sigma \quad (\sigma \in \mathfrak{S}_3, \ j \in \{1,2,3\})$

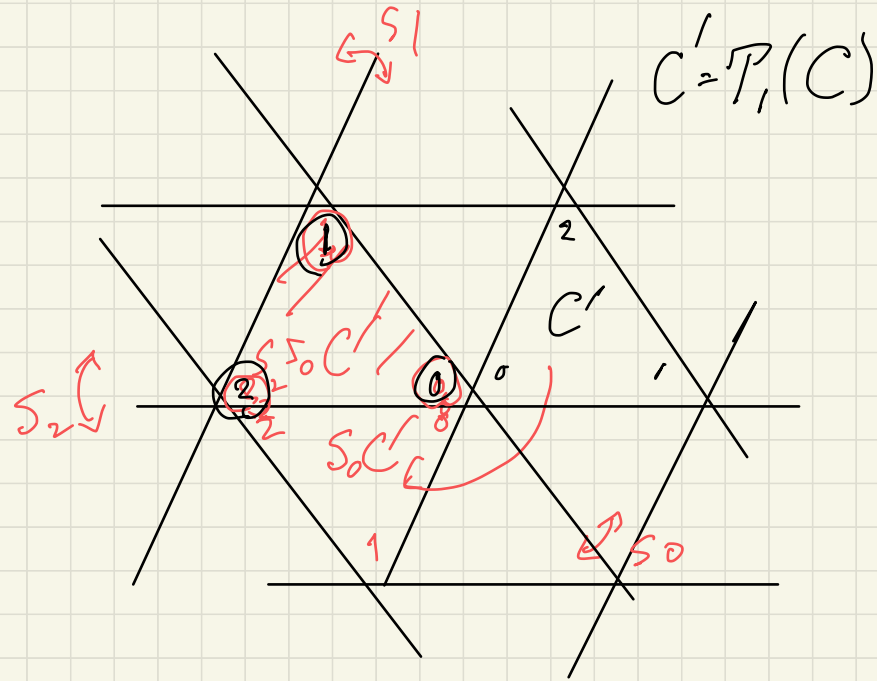
$T_i = T_{\varepsilon_i}$



$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2, \quad T_3 = s_2 s_1 \pi$

$(\omega = \pi^{-1})$





$$\underline{\underline{T_1}} = \underline{s_0 s_2} \pi = \underline{\pi s_2 s_1}$$

$$C' \xrightarrow{s_0} s_0 C' \xrightarrow{s_2} s_2 s_0 C' = C$$

$$1 = \pi^{-1} s_2 s_0 T_1 : C \to C \quad \begin{matrix} P_0 \mapsto P_0 \\ P_1 \mapsto P_1 \\ P_2 \mapsto P_2 \end{matrix} \quad \pi^{-1} s_2 s_0 (C') = \underline{\underline{C}}$$

$T_1 C$