## Symmetries of Painlevé equations: Lecture 12 (B4)

by M. Noumi [May 7, 2021]

## $5 \quad P_{\mathrm{VI}}$ and Umemura polynomials

## - References:

- G. Casale, L. Di Vizio, J.-P. Ramis: Volume à la mémoire de Hiroshi Umemura: "Équations de Painlevé et théories de Galois différentielles", Ann. Fac. Sci. Toulouse Math. 29 (2020), n ${ }^{\circ} 5$.
- M. Noumi: Notes on Umemura polynomials. Ann. Fac. Sci. Toulouse Math. 29 (2020), n ${ }^{\circ}$ 5, 1091-1118.


### 5.1 Hamiltonian system $H_{\text {IV }}$

The sixth Painlevé equation

$$
\begin{align*}
\left(P_{\mathrm{VI}}\right) \quad \frac{d^{2} y}{d t^{2}}= & \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{d y}{d t}\right)^{2}-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right) \frac{d y}{d t}  \tag{5.1}\\
& +\frac{y(y-1)(y-t)}{t^{2}\left(t^{2}-1\right)}\left(\alpha+\beta \frac{t}{y^{2}}+\gamma \frac{t-1}{(y-1)^{2}}+\delta \frac{t(t-1)}{(y-t)^{2}}\right)
\end{align*}
$$

for $y=y(t)$ is expressed as a Hamiltonian system

$$
\begin{equation*}
\left(H_{\mathrm{VI}}\right) \quad \frac{d q}{d t}=\frac{\partial H}{\partial p}, \quad \frac{d p}{d t}=-\frac{\partial H}{\partial q} \tag{5.2}
\end{equation*}
$$

with Hamiltonian $H=H(q, p ; t)$ defined by

$$
\begin{align*}
t(t-1) H= & p^{2} q(q-1)(q-t)-p\left\{\left(\alpha_{0}-1\right) q(q-1)+\alpha_{3} q(q-t)+\alpha_{4}(q-1)(q-t)\right\}  \tag{5.3}\\
& +\alpha_{2}\left(\alpha_{1}+\alpha_{2}\right)(q-t),
\end{align*}
$$

or explicitly,

$$
\begin{align*}
t(t-1) \frac{d q}{d t}= & 2 p q(q-1)(q-t)-\left(\alpha_{0}-1\right) q(q-1)-\alpha_{3} q(q-t)-\alpha_{4}(q-1)(q-t) \\
t(t-1) \frac{d p}{d t}= & -p^{2}((q-1)(q-t)+q(q-t)+q(q-1))  \tag{5.4}\\
& \quad+p\left\{\left(\alpha_{0}-1\right)(2 q-1)+\alpha_{3}(2 q-t)+\alpha_{4}(2 q-1-t)\right\}-\alpha_{2}\left(\alpha_{1}+\alpha_{2}\right) .
\end{align*}
$$

Here $\alpha_{0}, \alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}$ are complex parameters subject to $\alpha_{0}+\alpha_{1}+2 \alpha_{2}+\alpha_{3}+\alpha_{4}=1$. One can directly verify that the Hamiltonian system ( $H_{\mathrm{VI}}$ ) is equivalent to the sixth Painlevé equation ( $P_{\mathrm{VI}}$ ) for $y(t)=q(t)$ with parameters

$$
\begin{equation*}
\alpha=\frac{1}{2} \alpha_{1}^{2}, \quad \beta=-\frac{1}{2} \alpha_{4}^{2}, \quad \gamma=\frac{1}{2} \alpha_{3}^{2}, \quad \delta=\frac{1}{2}\left(1-\alpha_{0}^{2}\right) . \tag{5.5}
\end{equation*}
$$

## 5.2 $\quad P_{\mathrm{VI}}$ as the compatibility condition of a linear equation (Lax pair)

The Hamiltonian system ( $H_{\mathrm{VI}}$ ) arises from the monodromy preserving deformation

$$
\begin{equation*}
\left(\partial_{z}^{2}+a_{1}(z ; t) \partial_{z}+a_{2}(z ; t)\right) u(z ; t)=0, \quad \partial_{t} u(z ; t)=\left(b_{1}(z ; t) \partial_{z}+b_{2}(z ; t)\right) u(z ; t) \tag{5.6}
\end{equation*}
$$

of a second order Fuchsian differential equation for $u=u(z ; t)$ on $\mathbb{P}^{1}$ with four regular singular points and an apparent singularity, where $\partial_{z}=\partial / \partial z$ and $\partial_{t}=\partial / \partial t$. The first equation is assumed to be Fuchsian with Riemann scheme

$$
\left\{\begin{array}{ccccc}
z=0 & z=1 & z=t & z=q & z=\infty  \tag{5.7}\\
0 & 0 & 0 & 0 & \alpha_{2} \\
\alpha_{4} & \alpha_{3} & \alpha_{0} & 2 & \alpha_{1}+\alpha_{2}
\end{array}\right\}, \quad \alpha_{0}+\alpha_{1}+2 \alpha_{2}+\alpha_{3}+\alpha_{4}=1,
$$

and the singularity $z=q$ with characteristic exponents 0,2 , which may depend on $t$, is assumed to be non-logarithmic. Under these assumptions, the coefficients $a_{1}(z ; t)$ and $a_{2}(z ; t)$ are determined uniquely by $p=\operatorname{Res}_{z=q}\left(a_{2}(z ; t) d z\right)$ as

$$
\begin{align*}
& a_{1}(z ; t)=\frac{1-\alpha_{4}}{z}+\frac{1-\alpha_{3}}{z-1}+\frac{1-\alpha_{0}}{z-t}-\frac{1}{z-q}, \\
& a_{2}(z ; t)=\frac{1}{z(z-1)}\left(-\frac{t(t-1) H}{z-t}+\frac{q(q-1) p}{z-q}+\alpha_{2}\left(\alpha_{1}+\alpha_{2}\right)\right), \tag{5.8}
\end{align*}
$$

In particular the Hamiltonian is obtained as $H=-\operatorname{Res}_{z=t}\left(a_{2}(z ; t) d z\right)$.

### 5.3 Affine Weyl group symmetry

We introduce parameters $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \varepsilon_{4}$ such that

$$
\begin{equation*}
\alpha_{0}=1-\varepsilon_{1}-\varepsilon_{2}, \quad \alpha_{1}=\varepsilon_{1}-\varepsilon_{2}, \quad \alpha_{2}=\varepsilon_{2}-\varepsilon_{3}, \quad \alpha_{3}=\varepsilon_{3}-\varepsilon_{4}, \quad \alpha_{4}=\varepsilon_{3}+\varepsilon_{4} \tag{5.9}
\end{equation*}
$$

and regard $\varepsilon=\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \varepsilon_{4}\right)$ as the coordinates of the four dimensional affine space $V=\mathbb{C}^{4}$. We identify this parameter space $V$ with the Cartan subalgebra of the simple Lie algebra $\mathfrak{s o}(8)$ of type $D_{4}$, and regard the parameters $\alpha_{j}(j=0,1,2,3,4)$ as the simple affine roots of type $D_{4}^{(1)}$ identifying the null root $\delta=\alpha_{0}+\alpha_{1}+2 \alpha_{2}+\alpha_{3}+\alpha_{4}$ with the constant function 1 on $V$.


It has been known since the pioneering work of Kazuo Okamoto in 1980's (Studies on the Painlev'e equations) that the Hamiltonian system ( $H_{\mathrm{VI}}$ ) admits a group of Bäcklund transformations which is isomorphic to the extended affine Weyl group $\widetilde{W}\left(D_{4}^{(1)}\right)$ of type $D_{4}$. This group $\widetilde{W}\left(D_{4}^{(1)}\right)=$ $\left\langle s_{0}, s_{1}, s_{2}, s_{3}, s_{4} ; r_{1}, r_{3}, r_{4}\right\rangle$ is generated by simple reflections $s_{i}(i=0,1, \ldots, 4)$ attached to the simple affine roots $\alpha_{i}$ and the Dynkin diagram automorphisms $r_{1}, r_{3}$ and $r_{4}=r_{1} r_{3}$ corresponding to the permutations $\sigma_{1}=(01)(34), \sigma_{3}=(03)(14), \sigma_{4}=(04)(13)$ of $\{0,1,3,4\}$ : these generators are subject to the fundamental relations

$$
\begin{equation*}
s_{i}^{2}=1 \quad(i=0,1, \ldots, 4) ; \quad s_{i} s_{j}=s_{j} s_{i} \quad(i, j=0,1,3,4) ; \quad s_{i} s_{2} s_{i}=s_{2} s_{i} s_{2} \quad(i=0,1,3,4) \tag{5.11}
\end{equation*}
$$

Table 1: Fundamental Bäcklund transformations of $\left(H_{\mathrm{VI}}\right)$

|  | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $q$ | $p$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $s_{0}$ | $-\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}+\alpha_{0}$ | $\alpha_{3}$ | $\alpha_{4}$ | $q$ | $p-\frac{\alpha_{0}}{q-t}$ |
| $s_{1}$ | $\alpha_{0}$ | $-\alpha_{1}$ | $\alpha_{2}+\alpha_{1}$ | $\alpha_{3}$ | $\alpha_{4}$ | $q$ | $p$ |
| $s_{2}$ | $\alpha_{0}+\alpha_{2}$ | $\alpha_{1}+\alpha_{2}$ | $-\alpha_{2}$ | $\alpha_{3}+\alpha_{2}$ | $\alpha_{4}+\alpha_{2}$ | $q+\frac{\alpha_{2}}{p}$ | $p$ |
| $s_{3}$ | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}+\alpha_{3}$ | $-\alpha_{3}$ | $\alpha_{4}$ | $q$ | $p-\frac{\alpha_{3}}{q-1}$ |
| $s_{4}$ | $\alpha_{0}$ | $-\alpha_{1}$ | $\alpha_{2}+\alpha_{4}$ | $\alpha_{3}$ | $-\alpha_{4}$ | $q$ | $p-\frac{\alpha_{4}}{q}$ |
| $r_{1}$ | $\alpha_{1}$ | $\alpha_{0}$ | $\alpha_{2}$ | $\alpha_{4}$ | $\alpha_{3}$ | $\frac{t(q-1)}{q-t}$ | $-\frac{(q-t)\left((q-t) p+\alpha_{2}\right)}{t(t-1)}$ |
| $r_{3}$ | $\alpha_{3}$ | $\alpha_{4}$ | $\alpha_{2}$ | $\alpha_{0}$ | $\alpha_{1}$ | $\frac{t}{q}$ | $-\frac{q\left(q p+\alpha_{2}\right)}{t}$ |
| $r_{4}$ | $\alpha_{4}$ | $\alpha_{3}$ | $\alpha_{2}$ | $\alpha_{1}$ | $\alpha_{0}$ | $\frac{q-t}{q-1}$ | $\frac{(q-1)\left((q-1) p+\alpha_{2}\right)}{t-1}$ |

and

$$
\begin{equation*}
r_{i}^{2}=1 \quad(i=1,3) ; \quad r_{1} r_{3}=r_{3} r_{1} ; \quad r_{k} s_{i}=s_{\sigma_{k}(i)} r_{k} \quad(k=1,3: i=0,1,3,4) . \tag{5.12}
\end{equation*}
$$

The Hamiltonian system $\left(H_{\mathrm{VI}}\right)$ has eight fundamental Bäcklund transformations $s_{0}, s_{1}, s_{2}, s_{3}, s_{4}$ and $r_{1}, r_{3}, r_{4}$ as specified in Table 1 in terms of the action on the generators of the differential field $\mathcal{K}=\mathbb{C}(\alpha ; q, p, t)$ attached to $\left(H_{\mathrm{VI}}\right)$. Note also that these transformations preserve the Poisson bracket such that $\{p, q\}=1$.

We remark that, besides this transformation group $\widetilde{W}=\widetilde{W}\left(D_{4}^{(1)}\right) \subset \operatorname{Aut}\left(\mathcal{K}, \partial_{t}\right)$, the Hamiltonian system $\left(H_{\mathrm{VI}}\right)$ has birational canonical transformations corresponding to the permutation group $\mathfrak{S}_{\{0,1,3,4\}}$ of the indexing set $\{0,1,3,4\}$. We describe in Table 2 the birational canonical transformations $\sigma_{i j}$ corresponding to the transpositions $(i j) \in \mathfrak{S}_{\{0,1,3,4\}}$. The Bäcklund transformations $r_{1}, r_{3}, r_{4}$ are obtained from these birational canonical transformations as compositions $r_{1}=\sigma_{01} \sigma_{34}, r_{3}=\sigma_{03} \sigma_{14}, r_{4}=\sigma_{04} \sigma_{13}$. If we include these birational canonical transformations that represent $\mathfrak{S}_{\{0,1,3,4\}}$ as well, the symmetry group of $\left(H_{\mathrm{VI}}\right)$ extends to an affine Weyl group $W\left(F_{4}^{(1)}\right)$ of type $F_{4}$.

In the context of monodromy preserving deformation, the Bäcklund transformations $s_{0}, s_{1}, s_{3}, s_{4}$ except $s_{2}$ arise from gauge transformations $u(z ; t) \rightarrow g(z ; t) u(z ; t)$ by power functions in $z$ branching only at $z=0,1, t, \infty$. Also, the birational canonical transformations of Table 2 are obtained from the coordinate changes in $z$ by fractional linear transformations that permutes the singular points $\{0,1, t, \infty\}$. As fo $s_{2}$, Okamoto found this Bäcklund transformation by analyzing the diffrential equation to be satisfied by the $\tau$ function $\tau=\tau(t)$ defined by

$$
\begin{equation*}
H=\frac{d}{d t} \log \tau=\frac{1}{\tau} \frac{d \tau}{d t} ; \tag{5.13}
\end{equation*}
$$

$s_{2}$ is called the Okamoto transformation.

Table 2: Birational canonical transformations that realize $\mathfrak{S}_{\{0,1,3,4\}}$

|  | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $t$ | $\partial_{t}$ | $q$ | $p$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sigma_{01}$ | $\alpha_{1}$ | $\alpha_{0}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ | $1-t$ | $-\partial_{t}$ | $\frac{(1-t) q}{q-t}$ | $\frac{(q-t)\left((q-t) p+\alpha_{2}\right)}{t((t-1)}$ |
| $\sigma_{03}$ | $\alpha_{3}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{0}$ | $\alpha_{4}$ | $\frac{1}{t}$ | $-t^{2} \partial_{t}$ | $\frac{q}{t}$ | $t p$ |
| $\sigma_{04}$ | $\alpha_{4}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{0}$ | $\frac{t}{t-1}$ | $-(t-1)^{2} \partial_{t}$ | $\frac{q-t}{1-t}$ | $(1-t) p$ |
| $\sigma_{13}$ | $\alpha_{0}$ | $\alpha_{3}$ | $\alpha_{2}$ | $\alpha_{1}$ | $\alpha_{4}$ | $\frac{t}{t-1}$ | $-(t-1)^{2} \partial_{t}$ | $\frac{q}{q-1}$ | $-(q-1)\left((q-1) p+\alpha_{2}\right)$ |
| $\sigma_{14}$ | $\alpha_{0}$ | $\alpha_{4}$ | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{1}$ | $\frac{1}{t}$ | $-t^{2} \partial_{t}$ | $\frac{1}{q}$ | $-q\left(q p+\alpha_{2}\right)$ |
| $\sigma_{34}$ | $\alpha_{0}$ | $\alpha_{1}$ | $\alpha_{2}$ | $\alpha_{4}$ | $\alpha_{3}$ | $1-t$ | $-\partial_{t}$ | $1-q$ | $-p$ |

### 5.4 Typical classical solutions

When $\alpha_{2}=0$, the specialization $p=0$ implies the Riccati equation

$$
\begin{equation*}
t(t-1) q^{\prime}=-\left(\alpha_{0}-1\right) q(q-1)-\alpha_{3} q(q-t)-\alpha_{4}(q-1)(q-t), \quad{ }^{\prime}=d / d t \tag{5.14}
\end{equation*}
$$

Then, by the change of variables

$$
\begin{equation*}
q=\frac{t(1-t)}{\alpha_{1}} \frac{v^{\prime}}{v}, \quad v=(1-t)^{\alpha_{4}} u \tag{5.15}
\end{equation*}
$$

we obtain the Gauss hypergeometric differential equation

$$
\begin{equation*}
t(1-t) u^{\prime \prime}+(\gamma-(\alpha+\beta+1) t) u^{\prime}-\alpha \beta u=0 \tag{5.16}
\end{equation*}
$$

with $\alpha=1-\alpha_{3}, \beta=\alpha_{4}, \gamma=\alpha_{0}+\alpha_{4}$. In this case of $\left(H_{\mathrm{VI}}\right)$, we have five fundamental invariant divisors $p=0$ along $\alpha_{2}=0$ and $q=0,1, t, \infty$ along $\alpha_{j}=0(j=4,3,1,0)$, respectively. From each of those, we obtain a one-parameter family of Riccati solutions which are expressible in terms of the Gauss hypergeometric function. By the Bäckund transformations, we see that, along each reflection hyperplane of the affine Weyl group, $\left(H_{\mathrm{VI}}\right)$ has a one-paramter family of solutions which are expressed as rational functions of Gauss hypergeometric functions and their derivatives. (See Forrester-Witte (2004, Nagoya Math. J.) for the relation to the random matrix theory.)

At the fixed point of each diagram automorphism $r_{j}(j=1,3,4)$, there arise algebraic solutions with two continuous parameters. From the data of Table 1 , the condition of a fixed point of $r_{3}$ is given by

$$
\begin{equation*}
\alpha_{0}=\alpha_{3}, \alpha_{1}=\alpha_{4}, \quad t / q=q, \quad-q\left(q p+\alpha_{2}\right) / t=p \tag{5.17}
\end{equation*}
$$

Solving these equations, we see that, for the parameter values

$$
\begin{equation*}
\alpha=\left(\alpha_{0}, \alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right)=\left(a+\frac{1}{2}, b,-a-b, a+\frac{1}{2}, b\right) \quad(a, b \in \mathbb{C}) \tag{5.18}
\end{equation*}
$$

there exist simple algebraic solutions

$$
\begin{equation*}
(q(t), p(t))=\left(t^{\frac{1}{2}}, \frac{1}{2}(a+b) t^{-\frac{1}{2}}\right) . \tag{5.19}
\end{equation*}
$$

As we will see below, the class of algebraic solutions obtained from this seed solutions by Bäcklund transformations show a remarkable combinatorial property; the corresponding special polynomials ( $\phi$ factors) are called the Umemura polynomials. Typical algebraic solutions are obtained in this way by Bäcklund transformations from those at the fixed points of diagram automorphisms. In fact, $\left(H_{\mathrm{VI}}\right)$ has many other algebraic solutions; the classification of agebraic solutions of ( $H_{\mathrm{VI}}$ ) has been completed by the work of Lisovyy-Tykhyy (2014).

### 5.5 Umemura polynomials

We now consider the Bäcklund transformation obtained by the translation $T=s_{0} s_{2} s_{1} s_{4} s_{2} s_{0} r_{3} \in$ $\widetilde{W}\left(D_{4}^{(1)}\right)$ such that

$$
\begin{equation*}
\left(T^{n}\left(\alpha_{0}\right), T^{n}\left(\alpha_{1}\right), T^{n}\left(\alpha_{2}\right), T^{n}\left(\alpha_{3}\right), T^{n}\left(\alpha_{4}\right)\right)=\left(\alpha_{0}+n, \alpha_{1}, \alpha_{2}, \alpha_{3}-n, \alpha_{4}\right) \quad(n \in \mathbb{Z}) \tag{5.20}
\end{equation*}
$$

Applying $T^{n}(n \in \mathbb{Z})$ to the seed solution (5.17), we obtain a sequence of algebraic solutions ( $q_{n}, p_{n}$ ) with parameters

$$
\begin{equation*}
\left(\alpha_{0}, \alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right)=\left(a+\frac{1}{2}+n, b,-a-b, a+\frac{1}{2}-n, b\right) . \quad(n \in \mathbb{Z}) \tag{5.21}
\end{equation*}
$$

The corresponding $\tau$ functions $\tau_{n}(n \in \mathbb{Z})$ can be written in the form

$$
\begin{equation*}
\tau_{n}=\text { const. } x^{\lambda_{n}}(x+1)^{\nu_{n}}(x-1)^{\nu_{n}} U_{n}(x), \quad x=t^{\frac{1}{2}}, \tag{5.22}
\end{equation*}
$$

where $U_{n}(x)$ are polynomials in $x=t^{\frac{1}{2}}$ determined by the Toda equation

$$
\begin{align*}
U_{n+1} U_{n-1}= & \left(x^{2}-1\right)^{2}\left(x U_{n}^{\prime \prime} U_{n}+U_{n}^{\prime} U_{n}-x\left(U_{n}^{\prime}\right)^{2}\right)  \tag{5.23}\\
& -\left(a^{2}(x+1)^{2}-b^{2}(x-1)^{2}-(2 n-1)^{2} x\right) U_{n}^{2} \quad(n \in \mathbb{Z})
\end{align*}
$$

with initial values $U_{0}(x)=U_{1}(x)=1$. In spite of the rational nature of this recurrence formula, it turns out that $U_{n}(x)$ are in fact polynomials in $x ; U_{n}(x)$ are called the Umemura polynomials attached to the seed solution (5.17). We demonstrate below some computations of these Umemura polynomials.

$$
\begin{align*}
U_{0}= & 1, \quad U_{1}=1, \quad U_{2}=-(a-b)(a+b)+\left(1-2 a^{2}-2 b^{2}\right) x-(a-b)(a+b) x^{2} \\
U_{3}= & -(-1+a-b)(a-b)(1+a-b)(-1+a+b)(a+b)(1+a+b) \\
& -3(a-b)^{2}(a+b)^{2}\left(-5+2 a^{2}+2 b^{2}\right) x \\
& -3(a-b)(a+b)\left(6-14 a^{2}+5 a^{4}-14 b^{2}+6 a^{2} b^{2}+5 b^{4}\right) x^{2} \\
& +\left(9-38 a^{2}+58 a^{4}-20 a^{6}-38 b^{2}+60 a^{2} b^{2}-12 a^{4} b^{2}+58 b^{4}-12 a^{2} b^{4}-20 b^{6}\right) x^{3}  \tag{5.24}\\
& -3(a-b)(a+b)\left(6-14 a^{2}+5 a^{4}-14 b^{2}+6 a^{2} b^{2}+5 b^{4}\right) x^{4} \\
& -3(a-b)^{2}(a+b)^{2}\left(-5+2 a^{2}+2 b^{2}\right) x^{5} \\
& -(-1+a-b)(a-b)(1+a-b)(-1+a+b)(a+b)(1+a+b) x^{6}
\end{align*}
$$

Note that, if $f(x)$ is a polynomial in $x$ of degree $\leq d$, it can be written as $f(x)=F(x+1, x-1)$ by a unique homogenous polynomial $F(u, v)$ in $(u, v)$ of degree $d$. If we rewrite $U_{n}$ as a homogenous polynomial of $u=x+1$ and $v=x-1$ of degree $n(n-1)$, we see that $U_{n}$ can be expressed in a concise form, and that something more interesting is happening.

$$
\begin{align*}
U_{2} & =-\frac{1}{4}(2 a-1)(2 a+1)(x+1)^{2}+\frac{1}{4}(2 b-1)(2 b+1)(x-1)^{2} \\
& =\frac{1}{2^{2}}\left(-(c-1)(c+1) u^{2}+(d-1)(d+1) v^{2}\right), \tag{5.25}
\end{align*}
$$

where we have set $2 a=c$ and $2 b=d$. Similarly, we have

$$
\begin{align*}
& 2^{6} U_{3} \\
& =-(c-3)(c-1)^{2}(c+1)^{2}(c+3) u^{6}+3(c-3)(c-1)(c+1)(c+3)(d-1)(d+1) u^{4} v^{2}  \tag{5.26}\\
& \quad-3(c-1)(c+1)(d-3)(d-1)(d+1)(d+3) u^{2} v^{4}+(d-3)(d-1)^{2}(d+1)^{2}(d+3) v^{6} .
\end{align*}
$$

We now introduce bilateral shifted factorials

$$
\begin{equation*}
c_{k}=\prod_{i=0}^{k-1}(c-k+1+2 i), \quad d_{k}=\prod_{i=0}^{k-1}(d-k+1+2 i) \quad(k=0,1,2, \ldots) \tag{5.27}
\end{equation*}
$$

so that

$$
\begin{align*}
& c_{0}=1, \quad c_{1}=c, \quad c_{2}=(c-1)(c+1), \quad c_{3}=(c-2) c(c+2),  \tag{5.28}\\
& c_{4}=(c-3)(c-1)(c+1)(c+3), \ldots
\end{align*}
$$

Then we have

$$
\begin{align*}
& 2^{2} U_{2}=-c_{2} u^{2}+d_{2} v^{2} \\
& 2^{6} U_{3}=-c_{4} c_{2} u^{6}+3 c_{4} d_{2} u^{4} v^{2}-3 c_{2} d_{4} u^{2} v^{4}+d_{4} d_{2} v^{6} . \tag{5.29}
\end{align*}
$$

The coefficients $1,3,3,1$ are binomial coefficients. If we proceed to the next step,

$$
\begin{align*}
2^{12} U_{4}= & c_{6} c_{4} c_{2} u^{12}-6 c_{6} c_{4} d_{2} u^{10} v^{2}+15 c_{6} c_{2} d_{4} u^{8} v^{4}-10 c_{6} d_{4} d_{2} u^{6} v^{6} \\
& -10 c_{4} c_{2} d_{6} u^{6} v^{6}+15 c_{4} d_{6} d_{2} u^{4} v^{8}-6 c_{2} d_{6} d_{4} u^{2} v^{10}+d_{6} d_{4} d_{2} v^{12} \tag{5.30}
\end{align*}
$$

where the binomial coefficient $\binom{6}{3}=20$ has split into two factorized terms with the same monomial $u^{6} v^{6}$. Let's check one more:

$$
\begin{align*}
2^{20} U_{5}= & c_{8} c_{6} c_{4} c_{2} u^{20}-10 c_{8} c_{6} c_{4} d_{2} u^{18} v^{2}+45 c_{8} c_{6} c_{2} d_{4} u^{16} v^{4}-50 c_{8} c_{6} d_{4} d_{2} u^{14} v^{6} \\
& -70 c_{8} c_{4} c_{2} d_{6} u^{14} v^{6}+175 c_{8} c_{4} d_{6} d_{2} u^{12} v^{8}+35 c_{6} c_{4} c_{2} d_{8} u^{12} v^{8}-126 c_{8} c_{2} d_{6} d_{4} u^{10} v^{10} \\
& -126 c_{6} c_{4} d_{8} d_{2} u^{10} v^{10}+35 c_{8} d_{6} d_{4} d_{2} u^{8} v^{12}+175 c_{6} c_{2} d_{8} d_{4} u^{8} v^{12}-70 c_{6} d_{8} d_{4} d_{2} u^{6} v^{14}  \tag{5.31}\\
& -50 c_{4} c_{2} d_{8} d_{6} u^{6} v^{14}+45 c_{4} d_{8} d_{6} d_{2} u^{4} v^{16}-10 c_{2} d_{8} d_{6} d_{4} u^{2} v^{18}+d_{8} d_{6} d_{4} d_{2} v^{20} .
\end{align*}
$$

Seeing these formulas, one would expect that the general formula could be
with some positive integers $n_{I, J}$, where the summation is taken over all divisions of the set $\{2,4, \ldots, 2 n\}$ into the disjoint union of two subsets.

A remarkable fact about these Umemura polynomials $U_{n+1}$ is that each coefficient $n_{I, J}$ in (5.32) represents the dimension of an irreducible polynomial representation of $G L(n+1)$. Recall that the irreducible polynomial representations of the general linear group $G L(n+1)$ are parametrized by the partitions $\lambda$ with $l(\lambda) \leq n+1$, i.e. non-increasing sequences of nonnegative integers

$$
\begin{equation*}
\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n+1}\right), \quad \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n+1} \geq 0 . \tag{5.33}
\end{equation*}
$$

For each partition $\lambda$, we denote by $L_{G L(n+1)}(\lambda)$ the irreducible representation of highest weight $\lambda$. Then the dimension of $L_{\mathrm{GL}(n+1)}(\lambda)$ is given by the hook-length formula

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} L_{G L(n+1)}(\lambda)=\prod_{s \in \lambda} \frac{n+1+c_{\lambda}(s)}{h_{\lambda}(s)}, \tag{5.34}
\end{equation*}
$$

where for each box $s=(i, j)$ in the Young diagram $\lambda, c_{\lambda}(s)=j-i$ and $h_{\lambda}(s)=\lambda_{i}-i+\lambda_{j}^{\prime}-j+1$ denote the content and the hook-length of $s$ respectively.

Returning to the formula (5.32), let $(I, J)$ be a pair of subsets of $\{2,4, \ldots, 2 n\}$ such that $I \sqcup J=\{2,4, \ldots, 2 n\}$. We express the subsets $I$ and $J$ in the form $I=\left\{2 i_{1}, 2 i_{2}, \ldots, 2 i_{r}\right\}, J=$ $\left\{2 j_{1}, 2 j_{2}, \ldots, 2 j_{s}\right\}$ by two strict partitions $i_{1}>\cdots>i_{r}$ and $j_{1}>\cdots>j_{s}$ such that $\left\{i_{1}, \ldots, i_{r}\right\} \sqcup$ $\left\{j_{1}, \ldots, j_{s}\right\}=\{1, \ldots, n\}$. Then, to the strict partition $i_{1}>i_{2}>\cdots>i_{r}$ we attach a partition

$$
\begin{equation*}
\lambda_{I}=\left(i_{1}-1, \ldots, i_{r}-1 \mid i_{1}, i_{2}, \ldots, i_{r}\right) \tag{5.35}
\end{equation*}
$$

in terms of the Frobenius symbol. The Frobenius symbol $\lambda=\left(a_{1}, \ldots, a_{r} \mid b_{1}, \ldots, b_{r}\right)$ with $a_{1}>$ $\cdots>a_{r} \geq 0, b_{1}>\cdots>b_{r} \geq 0$, represents the partition $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots\right)$ such that $\lambda_{i}-i=a_{i}$ for $i=1, \ldots, r$, and $\lambda_{j}^{\prime}-j=b_{j}$ for $j=1, \ldots, r$. The observation made by Noumi-Okada-OkamotoUmemura (1998) was that

$$
\begin{equation*}
n_{I, J}=\operatorname{dim}_{\mathbb{C}} L_{G L(n+1)}\left(\lambda_{I}\right)=\operatorname{dim}_{\mathbb{C}} L_{G L(n+1)}\left(\lambda_{J}\right) \tag{5.36}
\end{equation*}
$$

As an example, let's look at the coefficient of $c_{8} c_{4} d_{6} d_{2} u^{12} v^{8}$ in (5.31) for $2^{20} U_{5}$. In this case, $n=4, I=\{8,4\}, r=2$. Since $\left(i_{1}, i_{2}\right)=(4,2)$, the corresponding partition is $\lambda_{I}=(3,1 \mid 4,2)=$ $(4,3,2,2,1)$.


Then the dimension of $L_{G L(5)}\left(\lambda_{I}\right)$ is computed by the hook-length formula (5.34) as

The conjecture (5.32), (5.36) was proved by Taneda (2001). On the basis of this work, KirillovTaneda (2002) also investigated a generalization of Umemura polynomials from the combinatorial point of view. In view of the work of Masuda (2003) on algebraic solutions of $P_{\mathrm{VI}}$ and the combinatorial viewpoint of Kirillov-Taneda, one can formulate a precise conjecture concerning combinatorial formula for Umemura polynomials $U_{m, n}(x)$ with two discrete parameters associated with a larger class of algebraic solutions of $P_{\mathrm{VI}}$.

